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For T.D.C. Part III
Paper-5
Gr- A

Example of a function whose double limit exists but repeated limits do not exist

Ex.4. *Show that the simultaneous (double limit) limit exists at the origin but the repeated limits do not exist for the function defined by* $f(x, y) = x \sin\frac{1}{y} + y \sin\frac{1}{x}$, $xy \neq 0$ *and* $f(0, 0) = 0$.

Soln. Let $\varepsilon > 0$ be given.

Here $\left| x \sin\frac{1}{y} + y \sin\frac{1}{x} - 0 \right| \leq |x| + |y|$; since $\left|\sin\frac{1}{x}\right| \leq 1$, $\left|\sin\frac{1}{y}\right| \leq 1$

$$= 2\sqrt{x^2 + y^2} < \varepsilon; \because |x| = \sqrt{x^2} < \sqrt{x^2 + y^2}$$

if $x^2 < \frac{\varepsilon^2}{4}$, $y^2 < \frac{\varepsilon^2}{4}$ i.e., if $|x| < \frac{\varepsilon}{2} = \delta$, $|y| < \frac{\varepsilon}{2} = \delta$.

Thus given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| x \sin\frac{1}{y} + y \sin\frac{1}{x} \right| < \varepsilon, \text{ whenever } |x| < \delta, |y| < \delta.$$

$$\therefore \quad \underset{(x, y) \to (0, 0)}{\text{Lt}} \left\{ x \sin\frac{1}{y} + y \sin\frac{1}{x} \right\} = 0.$$ Thus the double limit exists at $(0,0)$

But since $\lim_{x\to 0} f(x,y)$ & $\lim_{y\to 0} f(x,y)$ do not exist, therefore the ~~double~~ repeated limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ & $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ also do not exist.

Sufficient condition for the existence of Max of Min Value for the function of two variables

Sufficient conditions : Let $f(x, y)$ be a function of two variables x and y.

Let us denote $\dfrac{\partial^2 f}{\partial x^2}$ at (a, b) by r,

$\dfrac{\partial^2 f}{\partial x \partial y}$ or $\dfrac{\partial^2 f}{\partial y \partial x}$ at (a, b) by s;

and $\dfrac{\partial^2 f}{\partial y^2}$ at (a, b) by t.

We are going to show that if $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ at (a, b) and higher partial derivatives exi in the neighbourhood of (a, b), then $f(x, y)$ will have

(i) a maximum at (a, b) if $r < 0$ and $rt - s^2 > 0$,

(ii) a minimum at (a, b) if $r > 0$ and $rt - s^2 > 0$,

(iii) neither a maximum nor a minimum at (a, b) if $rt - s^2 < 0$ and $r \neq 0$;

(iv) if $rt - s^2 = 0$, the case is doubtful and needs further consideration.

It may be noted that if $rt - s^2 > 0$, then $r \neq 0$. (for in that case $-s^2$ becomes > 0, which is not true).

Proof. By Taylor's theorem (with remainder after three terms), we obtain

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a + \theta h, b + \theta k) \text{ where } 0 < \theta < 1$$

$$= f(a, b) + 0 + \frac{1}{2!}\left\{h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right\} f(a, b) + \rho,$$

$$\text{since } \frac{\partial}{\partial x} f(a, b) = 0, \frac{\partial}{\partial y} f(a, b) = 0$$

and where ρ is of third degree in h and k.

$$\Rightarrow \quad f(a+h, b+k) - f(a, b)$$

$$= \frac{1}{2!}\left\{h^2\frac{\partial^2}{\partial x^2} f(a, b) + 2hk\frac{\partial^2}{\partial x \partial y} f(a, b) + k^2\frac{\partial^2}{\partial y^2} f(a, b)\right\} + \rho$$

$$= \frac{1}{2}\{h^2 r + 2hks + k^2 t\} + \rho.$$

Obviously for sufficiently small values of h, k the sign of $f(a+h, b+k) - f(a, b)$ is determined by $\frac{1}{2}\{h^2 r + 2hks + k^2 t\}$ i.e., by $h^2 r + 2hks + k^2 t$.

In other words, $h^2 r + 2hks + k^2 t$ is the dominant expression.

Now, ignoring higher powers of h and k and hence ρ, we have approximately

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(h^2 r + 2hks + k^2 t)$$

$$= \frac{1}{2r}(h^2 r^2 + 2hkrs + k^2 rt)$$

$$= \frac{1}{2r}\{(hr + ks)^2 + (rt - s^2)k^2\}$$

Now, we discuss the sign of $f(a+h, b+k) - f(a, b)$ vis-a-vis the sign of

$$\frac{1}{2r}\{(hr + ks)^2 + (rt - s^2)k^2\}.$$

Accordingly, we discuss the following three cases.

Case I. Let $rt - s^2 > 0$.

In this case, neither r nor t can be zero.

Since $rt - s^2 > 0$, we find that $(rh + ks)^2 + (rt - s^2)k^2$ is always positive, except when $rh + sk = 0$, $k = 0$, i.e., when $h = 0$, $k = 0$.

Thus, we see that $rh^2 + 2shk + tk^2$ always retains the same sign as that of r.

Thus $f(a+h, b+k) - f(a, b) > 0$ if $r > 0$

and $f(a+h, b+k) - f(a, b) < 0$ if $r < 0$.

That is, $f(a, b)$ will be a minimum if $r > 0$,

and $f(a, b)$ will be a maximum if $r < 0$.

Hence, if $rt - s^2 > 0$ and if

(i) $r < 0$, $f(x, y)$ will be a maximum at (a, b)

(ii) $r > 0$, $f(x, y)$ will be a minimum at (a, b).

Case II. Let $rt - s^2 < 0$.

Firstly, we suppose that $r \neq 0$.

We write $rh^2 + 2shk + tk^2 = \frac{1}{r}\{(rh + sk)^2 + (rt - s^2)k^2\}$.

Since $rt - s^2 < 0$, we see that this expression sometimes takes up positive signs and sometimes takes up negative signs except when $k = 0$ and $rh + sk = 0$.

Thus in this case $f(a, b)$ is not an extreme value.

The proof is similar when $t \neq 0$.

In case, $r = 0$ as well as $t = 0$, we have $rh^2 + 2shk + tk^2 = 2shk$ so that the expression assumes values with different signs and consequently $f(a, b)$ is not an extreme value.

Case III. Let $rt - s^2 = 0$.

Suppose that $r \neq 0$.

Then we have $rh^2 + 2shk + tk^2 = \frac{1}{r}(rh + sk)^2 + (rt - s^2)k^2 = \frac{(rh + sk)^2}{r}$.

Here the expression $rh^2 + 2shk + tk^2$ becomes zero when $rh + sk = 0$, so that the nature of the sign of $f(a+h, b+k) - f(a, b)$ depends upon the consideration of ρ.

The case is, therefore doubtful.

If now $r = 0$, then because of the condition $rt - s^2 = 0$, we must have $s = 0$. Therefore $rh^2 + 2shk + tk^2 = tk^2$ so that the expression is zero when $k = 0$ whatever h may be.

The case is again doubtful.

Examples of functions which possess partial derivatives but are not differentiable

▸ **Ex.3.** *Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point $(0, 0)$ but f_x and f_y both exist at $(0, 0)$.*

Soln. We have,

$$f_x(0, 0) = \underset{h \to 0}{\text{Lt}}\left[\frac{f(0+h, 0) - f(0, 0)}{h}\right] = \underset{h \to 0}{\text{Lt}}\left[\frac{0-0}{h}\right] = 0$$

$$f_y(0, 0) = \underset{k \to 0}{\text{Lt}}\left[\frac{f(0, 0+k) - f(0, 0)}{k}\right] = \underset{k \to 0}{\text{Lt}}\left[\frac{0-0}{k}\right] = 0.$$

Thus f_x and f_y both exist at the origin and have the value 0.

Now, we need to show that $f(x, y)$ is not differentiable at $(0, 0)$.

If $f(x, y)$ were differentiable at $(0, 0)$, then we should have

$$f(0+h, 0+k) - f(0, 0) = Ah + Bk + h\phi(h, k) + k\psi(h, k) \quad \text{...(1)}$$

where $A = f_x(0, 0) = 0, B = f_y(0, 0) = 0$

and $\quad \underset{\substack{h \to 0 \\ k \to 0}}{\text{Lt}} \phi(h, k) = 0 = \underset{\substack{h \to 0 \\ k \to 0}}{\text{Lt}} \psi(h, k).$

(1) gives $\sqrt{|hk|} - 0 = h\phi(h, k) + k\psi(h, k)$

which $\Rightarrow \underset{(h, k) \to (0, 0)}{\text{Lt}} \sqrt{|hk|} = \underset{(h, k) \to (0, 0)}{\text{Lt}} [h\phi(h, k) + k\psi(h, k)]$

$$\Rightarrow \underset{(h, k) \to (0, 0)}{\text{Lt}}\left[\frac{(|hk|)^{1/2}}{h}\right] = \underset{(h, k) \to (0, 0)}{\text{Lt}}\left[\phi(h, k) + \frac{k}{h}\psi(h, k)\right] \quad \text{...(2)}$$

Putting $k = mh$ (so that as $h \to 0, k \to 0$) in (2), we get $\underset{(h, k) \to (0, 0)}{\text{Lt}} [|m|^{1/2}] = 0$

which is not true since m may have any value not necessarily zero.

Hence $f(x, y)$ is not differentiable at $(0, 0)$.