Triangular Form:- If $T: V \to V$ is a linear transformation on V over F, then the matrix of T in the basis $(v_1, v_2, \ldots v_n)$ of V is triangular if

$$T(v_1) = a_{11} v_1$$
$$T(v_2) = a_{21} v_1 + a_{22} v_2$$
$$T(v_3) = a_{31} v_1 + a_{32} v_2 + a_{33} v_3$$
$$\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots$$
$$T(v_n) = a_{n1} v_1 + a_{n2} v_2 + \cdots\cdots + a_{nn} v_n$$

Theorem - If $T \in A(V)$ has all its characteristic roots in F, then there is a basis of V in which the matrix of T is triangular.

Proof: We prove the theorem by induction on the dimension of V.

Let $\dim V = 1$, then every matrix representation of T is a matrix of order 1×1, which is trivially triangular. i.e $[a_{11}]$

Suppose that the theorem is true for all vectors spaces over F of dimension $n-1$.

Let $\dim V = n > 1$, Since T has all its characteristic roots in F.

Let $\lambda_1 \in F$ be a characteristic/ eigen root of T. Then there exists a non-zero eigen vector v_1 corresponding to λ_1 such that

$$T(v_1) = a_{11} v_1 .$$

Let W be the one-dimensional subspace of V spanned by v_1 and is T-invariant.

let $\tilde{V} = V/W$ then

$$\dim \tilde{V} = \dim V - \dim W$$
$$\dim \tilde{V} = n - 1$$

This theorem T induces a linear transformation $\tilde{T}$ on $\tilde{V}$ whose minimal polynomial divides the minimal polynomial of T. Therefore, all the roots of the minimal polynomial of $\tilde{T}$, being roots of the minimal polynomial of T must lie in F. Thus $\tilde{V}$ and $\tilde{T}$ satisfy the hypothesis of the theorem.

Since $\dim \tilde{V} = n-1$, then by induction hypothesis there is basis $(\tilde{v}_2, \tilde{v}_3, \cdots \tilde{v}_n)$ of $\tilde{V}$. such that

$$\tilde{T}(\tilde{v}_2) = a_{22}\tilde{v}_2$$
$$\tilde{T}(\tilde{v}_3) = a_{32}\tilde{v}_2 + a_{33}\tilde{v}_3$$
$$\cdots\cdots \quad \cdots\cdots$$
$$\cdots\cdots$$
$$\tilde{T}(\tilde{v}_n) = a_{n2}\tilde{v}_2 + a_{n3}\tilde{v}_3 + \cdots - a_{nn}\tilde{v}_n$$

Now let $(v_2, v_3, \cdots v_n)$ be the vectors/elements of V which belong to the cosets $\tilde{v}_2, \tilde{v}_3, \cdots \tilde{v}_n$ respectively i.e

$$\tilde{v}_i = v_i + W.$$

Then $(v_1, v_3 \cdots v_n)$ basis of V.

Since $\tilde{T}(\tilde{v}_2) = a_{22}\tilde{v}_2$

$$\tilde{T}(v_2+W) = a_{22}(v_2+W)$$

$$\tilde{T}(v_2)+W = a_{22}v_2+W$$

$$\Rightarrow \quad \tilde{T}(v_2) - a_{22}v_2 \in W$$

$$T(v_2) - a_{22}v_2 \in W$$

But W is spanned by v_1, so

$$T(v_2) - a_{22}v_2 = a_{21}v_1$$

$$T(v_2) = a_{21}v_1 + a_{22}v_2$$

Similarly for $\tilde{v}_3, \tilde{v}_4, \cdots \tilde{v}_n$ we have

$$T(v_i) = a_{i1}v_1 + a_{i2}v_2 + \cdots a_{in}v_n$$

Thus. $T(v_1) = a_{11}v_1$

$$T(v_2) = a_{21}v_1 + a_{22}v_2$$

$$T(v_3) = a_{31}v_1 + a_{32}v_2 + a_{33}v_3$$

$$\cdots$$

$$\cdots$$

$$T(v_n) = a_{n1}v_1 + a_{n2}v_2 + \cdots a_{nn}v_n$$

Hence the matrix of T in the basis $(v_1, v_2, \cdots v_n)$ is triangular.