Theorem - If a subspace W of V is T-invariant Then T has a block matrix representation $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where A is a matrix representative of the restriction $\tilde{T}$ of T to W.

Proof - Let $\{w_1, w_2 \cdots w_r\}$ be the basis of W. Then this basis is extended to a basis $\{w_1, w_2 \cdots w_r, v_1, v_2 \cdots v_s\}$ of V.

Since W is T-invariant, So we have

$$\tilde{T}(w_i) = T(w_i) \quad \text{for } i = 1, 2 \cdots r$$

$$\Rightarrow \tilde{T}(w_1) = T(w_1) = a_{11}w_1 + a_{12}w_2 + \cdots + a_{1r}w_r$$

$$\tilde{T}(w_2) = T(w_2) = a_{21}w_1 + a_{22}w_2 + \cdots + a_{2r}w_r$$

$$\cdots$$

$$\cdots$$

$$\tilde{T}(w_r) = T(w_r) = a_{r1}w_1 + a_{r2}w_2 + \cdots \cdots + a_{rr}w_r$$

Also for basis of V.

$$T(v_1) = b_{11}w_1 + b_{12}w_2 + \cdots b_{1r}w_r + c_{11}v_1 + c_{12}v_2 + \cdots c_{1s}v_s$$

$$T(v_2) = b_{21}w_1 + b_{22}w_2 + \cdots b_{2r}w_r + c_{21}v_1 + c_{22}v_2 + \cdots c_{2s}v_s$$

$$\cdots$$

$$\cdots$$

$$T(v_s) = b_{s1}w_1 + b_{s2}w_2 + \cdots b_{sr}w_r + c_{s1}v_1 + c_{s2}w_2 + \cdots c_{ss}v_s$$

The coefficient matrix from the system of L eqn is

$$A = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1r} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & & a_{2r} & 0 & 0 & \cdots & 0 \\ \cdots & & & & & & & \\ \cdot & \cdots & & & \cdot & & \cdots & \cdot \\ a_{r1} & a_{r2} & & a_{rr} & 0 & 0 & 0 \cdots & 0 \\ \hline b_{11} & b_{12} & \cdots & b_{1r} & c_{11} & c_{12} & \cdots & c_{1s} \\ b_{21} & b_{22} & \cdots & b_{2r} & c_{21} & c_{22} & \cdots & c_{2s} \\ \cdot & \cdots & & & & & & \\ \cdot & \cdots & & & \cdot & & & \\ b_{s1} & b_{s2} & \cdots & b_{sr} & c_{s1} & c_{s2} & \cdots & c_{ss} \end{array}\right]$$

Then the matrix representation of T with r.t. basis of W & V. is

$$[T]_B = A^T = \left[\begin{array}{cccc|cccc} a_{11} & a_{21} & \cdots & a_{r1} & b_{11} & b_{21} & \cdots & b_{s1} \\ a_{12} & a_{22} & \cdots & a_{r2} & b_{11} & b_{22} & \cdots & b_{s2} \\ \cdots & & \cdots & & \cdots & & & \\ \cdots & & \cdots & & \cdots & & & \\ a_{1r} & a_{2r} & \cdots & a_{rr} & b_{1r} & b_{2r} & \cdots & b_{sr} \\ \hline 0 & 0 & \cdots & 0 & c_{11} & c_{21} & \cdots & c_{s1} \\ 0 & 0 & \cdots & 0 & c_{12} & c_{22} & \cdots & c_{s2} \\ \cdots & & & & \cdots & & & \\ \cdots & & & & \cdots & & & \\ 0 & 0 & \cdots & 0 & c_{1s} & c_{2s} & \cdots & c_{ss} \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]$$

where $A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ a_{12} & a_{22} & \cdots & a_{r2} \\ \cdots & & & \\ a_{1r} & a_{2r} & \cdots & a_{rr} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{s1} \\ b_{21} & b_{22} & \cdots & b_{s2} \\ \cdots & & & \\ b_{1r} & b_{2r} & \cdots & b_{sr} \end{bmatrix}$

$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \end{bmatrix}$ $C = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{s1} \\ c_{12} & c_{22} & \cdots & c_{s2} \\ c_{1s} & \cdots & & c_{ss} \end{bmatrix}$