Theorem - Let $T: U \to V$ be a linear transformation and rank$(T) = r$. Then the bases of U and that of V such that the matrix representation of T has the form $A = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$

where I is the r-square Identity matrix.

Proof Let the $\dim U = m$ and $\dim V = n$

let W be the kernel of T and Im(T) the image of T.

Since the rank of T is r, therefore the dimension of the kernel of T i.e $\dim W = m-r$

Let $\{w_1, w_2, \cdots w_{m-r}\}$ be basis of W. So it can be extended to form a basis of U.

let this extension be

$$\{v_1, v_2, v_3 \cdots v_n, w_1, w_2 \cdots w_{m-r}\}$$

Now $u_1 = T(v_1)$, $u_2 = T(v_2) \cdots u_r = T(v_r)$

Thus the set $\{u_1, u_2, \cdots u_r\}$ forms a basis of Im(T), again this basis is extended to form a basis of V. Then this extension be

$$\{u_1, u_2 \cdots u_r, u_{r+1} \cdots u_n\}$$

We observe that

$$T(v_1) = u_1 = 1u_1 + 0u_2 + \cdots 0u_r + \cdots 0u_n$$

$$T(v_2) = u_2 = 0u_1 + 1u_2 + \cdots 0u_r + \cdots 0u_n$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$T(v_r) = u_r = 0u_1 + 0u_2 + \cdots 1u_r + \cdots 0u_n$$

$$T(w_1) = 0 = 0u_1 + 0u_2 + \cdots 0u_r + \cdots 0u_n$$
$$T(w_2) = 0 = 0u_1 + 0u_2 + \cdots 0u_r + \cdots 0u_n$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$T(w_{m-r}) = 0 = 0u_1 + 0u_2 + \cdots 0u_r + \cdots 0u_n$$

Thus the matrix representation of T is

$$A = \left[\begin{array}{ccccc|ccc} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]_{n \times n}$$

$r \times r$

or

$$A = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$$

[The matrix A is known as normal or canonical form]