05/06/2021

Theorem - If $\{u_1, u_2 \cdots u_n\}$ is an orthonormal basis of a subspace M of an inner product space $V(F)$ and if $\{v_1, v_2 \cdots v_m\}$ is an orthonormal basis of $M^\perp$ then show that

$\{u_1, u_2 \cdots u_n, v_1, v_2 \cdots v_m\}$ is an orthonormal basis of V.

Proof Let $\{u_1, u_2 \cdots u_n\}$ be an orthonormal basis of a subspace M of an inner product space $V(F)$ so that

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Here i and j take the values $1, 2, \cdots n$.

Also let $\{v_1, v_2, \cdots v_m\}$ be an orthonormal basis of $M^\perp$, so that

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Here $i, j = 1, 2, \cdots m$

To prove that $B = \{u_1, u_2 \cdots u_n, v_1, v_2 \cdots v_m\}$ is an orthonormal basis of V.

$M, M^\perp \subset V \Rightarrow u_i, v_j \in V$ for $i = 1, 2, \cdots n$
$j = 1, 2, \cdots m$

any $v \in M^\perp \Rightarrow \langle v, u \rangle = 0 \ \forall u \in M$

$\Rightarrow \langle v, u_i \rangle = 0$ for $i = 1, 2 \cdots n$

This is true for every $v \in M^\perp$.

$\because \langle v_j, u_i \rangle = 0$ for $i = 1, 2, \cdots n$ and $j = 1, 2, \cdots m$

Thus $\langle u_i, u_j \rangle = \delta_{ij}$, $\langle v_i, v_j \rangle = \delta_{ij}$

$\Rightarrow \langle u_i, v_j \rangle = 0 \;\; \forall \; i$ and j

Here $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

This $\Rightarrow B = \{u_1, u_2 \cdots u_n, v_1, v_2 \cdots v_m\}$ is an orthonormal set.

B is Linear independent

We know that $V = M \oplus M^{\perp}$

$\dim V = \dim M + \dim M^{\perp}$

$\dim V = n + m$

Thus B is L.I subset of V and number of vectors in B is $n + m = \dim V$.

This $\Rightarrow B$ is basis of V. But B is an orthonormal set.

Hence B is orthonormal basis of V.