Theorem - The kernel of linear transformation $T: V \to W$ is a subspace of the domain V.

Proof We know that $\ker(T)$ is a non-empty subset of V. So, we can show that $\ker(T)$ is a subspace of V ie $\ker(T) \subset V$ by closed conditions under vectors addition and scalar multiplication.

Let u, v be vectors in the kernel of T.

Then

$$T(u+v) = T(u) + T(v)$$
$$= 0 + 0$$
$$T(u+v) = 0$$

$\therefore \Rightarrow u+v$ is in kernel of T.

Moreover, if α is any scalar then

$$T(\alpha u) = \alpha T(u)$$
$$= \alpha 0$$
$$= 0$$

which implies that αu is in the kernel.

Hence kernel is a subspace of V.

Theorem If T is homomorphism of $T: V \to W$

then (i) $T(0) = 0$ where $0 \in V$, $0 \in W$

(ii) $T(-x) = -T(x) \quad \forall\, x \in V$

Proof for (i) $T(0) = 0$

we have

$$0 + 0 = 0$$

$$\Rightarrow T(0+0) = T(0)$$

$$T(0) + T(0) = T(0) \qquad T \text{ is L.T}$$

$$T(0) + T(0) = T(0) + 0 \quad (\text{holds in } W)$$

$$T(0) = 0 \qquad \text{by cancellation Law}$$

Hence $T(0) = 0$

(ii) for (ii) $T(-x) = -T(x)$

We have $x + (-x) = 0$

$$T(x + (-x)) = T(0)$$

$$T(x) + T(-x) = T(0)$$

$$T(x) + T(-x) = 0 \qquad \text{by (i)}$$

$$T(-x) = -T(x)$$