Q16 Find at what points on the curve $\frac{x^3}{9}+\frac{y^3}{6}=xy$ the tangent is parallel to one of the co-ordinate axes.

Sol. $$\frac{x^3}{9}+\frac{y^3}{6}=xy \quad \text{—} (1)$$

Diff. w.r.to x

$$3\frac{x^2}{9}+3\frac{y^2}{6}\cdot\frac{dy}{dx}=y+x\frac{dy}{dx}$$

$$\left(3\frac{y^2}{6}-x\right)\frac{dy}{dx}=y-\frac{3x^2}{9} \quad \text{—} (11)$$

The Eqn of tantgent at (x_1, y_1)

$$(y-y_1)=\frac{dy}{dx}(x-x_1)$$

Tangent is // to x axis.

$$\Rightarrow \frac{dy}{dx}=0$$

From (11)

$$\left(\frac{3y^2}{6}-x\right).0=y-\frac{3x^2}{9}$$

$$y=\frac{3x^2}{9}$$

Putting in (1)

$$\Rightarrow \frac{x^3}{a} + \frac{27x^6}{a^3 b} = x \cdot \frac{3x^2}{a}$$

$$\frac{1}{a} + \frac{27}{b}\frac{x^3}{a^3} = \frac{3}{a}$$

$$27\frac{x^3}{a^3 b} = \frac{2}{a}$$

$$27x^3 = 2ba^2$$

$$x^3 = \frac{2}{27} ba^2$$

$$x = \frac{(2)^{1/3}}{3} \cdot b^{1/3} a^{2/3}$$

$$y = \frac{3}{a} \cdot \frac{(2)^{2/3}}{9} \cdot b^{2/3} \cdot a^{4/3}$$

$$y = \frac{1}{3} \cdot (2)^{2/3} b^{2/3} \cdot a^{1/3}$$

$$(x, y) \Rightarrow \left(\frac{1}{3} \cdot 2^{1/3} \cdot b^{1/3} a^{2/3}, \frac{1}{3} 2^{2/3} b^{2/3} \cdot a^{1/3}\right)$$

Q.21 Find the equation of the tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ at the point $(a\cos^3\theta, a\sin^3\theta)$. If p and Q be the points corresponding to θ, $\pi/2+\theta$ and the tangents at these points meet at T, prove that $OP^2+OQ^2+3OT^2=$ Const

Sol. give curve $x^{2/3}+y^{2/3}=a^{2/3}$ —— ①

point p corresponding to θ.

$p(a\cos^3\theta, a\sin^3\theta)$

and point Q corresponding to $\theta \to \pi/2+\theta$

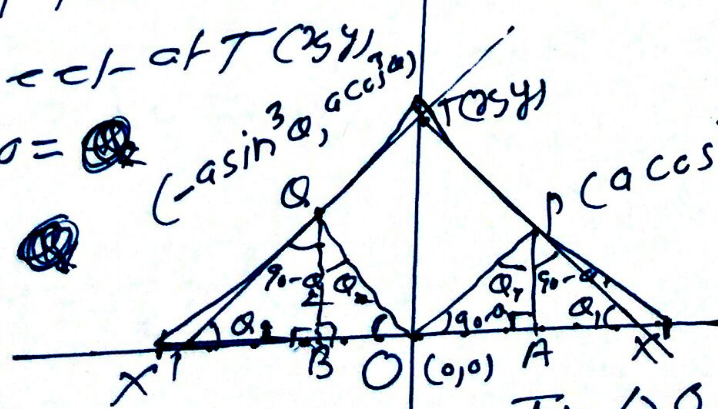
$Q(a\cos^3(\pi/2+\theta), a\sin^3(\pi/2+\theta))$

$Q(-a\sin^3\theta, a\cos^3\theta)$

The tangent XT touch the curve at p and tangent X'T touch at Q point and these tangents meet at T(x,y)

Angle $\angle TX'O=\theta$

and $\angle TXO=\theta$



In $\triangle OBQ$

$\angle B=90°$

$OB=-a\sin^3\theta$

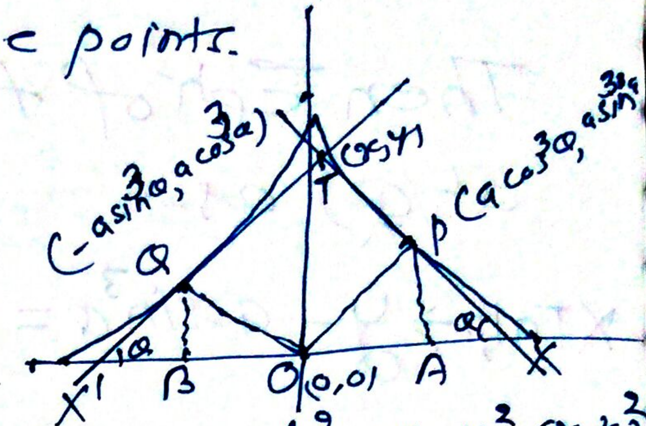
$QB= a\cos^3\theta$

In $\triangle OAP$, $\angle A=90°$

$OA=a\cos^3\theta$

$AP=a\sin^3\theta$

The co-ordinates of the points.



$O \to (0,0)$

$p \to (a\cos^3\theta, a\sin^3\theta)$

$Q \to (-a\sin^3\theta, a\cos^3\theta)$

Distance b/w two points

$d^2 = (y_2-y_1)^2 + (x_2-x_1)^2$

$Op^2 = (a\cos^3\theta)^2 + (a\sin^3\theta)^2$

$Op^2 = a^2\cos^6\theta + a^2\sin^6\theta \Rightarrow a^2[\cos^6\theta + \sin^6\theta]$ — (II)

Similarly

$OQ^2 = (-a\sin^3\theta)^2 + (a\cos^3\theta)^2$

$= a^2\sin^6\theta + a^2\cos^6\theta$

$= a^2[\sin^6\theta + \cos^6\theta]$ — (III)

OQ^2

To find the Equations of the tangents at p and Q from curve $x^{2/3} + y^{2/3} = a^{2/3}$

Diff. w.r.t. x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$\frac{dy}{dx}$ at $p(a\cos^3\theta, a\sin^3\theta) \Rightarrow \left(\frac{dy}{dx}\right)_{at\,p} = -\left(\frac{a\sin^3\theta}{a\cos^3\theta}\right)^{1/3}$

$= \left(\frac{dy}{dx}\right)_{at\,p} = -\frac{\sin\theta}{\cos\theta}$

Similarly $\frac{dy}{dx}$ at $Q(-a\sin^3\theta, a\cos^3\theta) \Rightarrow \left(\frac{dy}{dx}\right)_{at\,Q} = \frac{\cos\theta}{\sin\theta}$

Then Eqn's of tangents (XT at P and X'T at Q) as-

XT at P → $y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

XT Eqn at P ⇒

$$\cos\theta\, y - a\cos\theta\sin^3\theta = -\sin\theta\, x + a\sin\theta\cos^3\theta$$

$$\cos\theta\, y + \sin\theta\, x = a\sin\theta\cos^3\theta + a\cos\theta\sin^3\theta$$

$$= a\sin\theta\cos\theta(\cos^2\theta + \sin^2\theta)$$

$$\cos\theta\, y + \sin\theta\, x = a\sin\theta\cos\theta \quad \text{— (IV)}$$

Now X'T Eqn at Q

$$y - a.\cos^3\theta = \frac{\cos\theta}{\sin\theta}(x + a\sin^3\theta)$$

$$\sin\theta\, y - a\sin\theta\cos^3\theta = \cos\theta\, x + a\cos\theta\sin^3\theta$$

$$\sin\theta\, y - \cos\theta\, x = a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$\sin\theta\, y - \cos\theta\, x = a\sin\theta\cos\theta \quad \text{— (V)}$$

from Eqn (IV) and (V)

$\cos\theta * (\text{iv})$ and $\sin\theta * (\text{V})$ and adding.

$$\cos^2\theta\, y + \sin^2\theta\, y = a\sin\theta\cos^2\theta + a\sin^2\theta\cos\theta$$

$$y[\cos^2\theta + \sin^2\theta] = a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$

$$y.1 = a\sin\theta.\cos\theta(\sin\theta + \cos\theta)$$

$$y = a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$

Putting value of y in Eqn (IV)

$$\cos\alpha \cdot a\sin\alpha\cos\alpha(\sin\alpha+\cos\alpha)+\sin\alpha\, x = a\sin\alpha\cos\alpha$$

$$x = a\cos\theta - a\cos^2\alpha(\sin\alpha+\cos\theta)$$

$$= a\cos\alpha[1-\cos\alpha(\sin\alpha+\cos\alpha)]$$

$$= a\cos\theta[1-\sin\alpha\cos\theta-\cos^2\alpha]$$

$$= a\cos\alpha[\sin^2\alpha-\sin\alpha\cos\alpha]$$

$$x = a\sin\alpha\cos\alpha(\sin\alpha-\cos\alpha)$$

$$y = a\sin\alpha\cos\alpha(\sin\alpha+\cos\alpha)$$

The distance b/w O & T

$$OT^2 = [\underbrace{a\sin\alpha\cos\alpha(\sin\alpha+\cos\alpha)}_{y_2} - \underbrace{a}_{y_1}]^2 + [\underbrace{a\sin\alpha\cos\alpha(\sin\alpha-\cos\alpha)}_{x_2} - \underbrace{0}_{x_1}]^2$$

$$OT^2 = a^2\sin^2\alpha\cos^2\alpha[(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2]$$

$$= a^2\sin^2\alpha\cos^2\alpha[\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha+\sin^2\alpha+\cos^2\alpha-2\sin\alpha\cos\alpha]$$

$$OT^2 = 2a^2\sin^2\alpha\cos^2\alpha$$

$$3OT^2 = 6a^2\sin^2\alpha\cos^2\alpha \quad \text{—— (6)}$$

from Eqn (II), (III) and (6)

Adding Eqn (II), (III) and ⑥

$$OP^2 + OQ^2 + 3OT^2 = a^2[\sin^6\alpha + \cos^6\alpha] + a^2[\sin^6\alpha + \cos^6\alpha] + 6a^2\sin^2\alpha\cos^2\alpha$$

$$= 2a^2(\sin^6\alpha + \cos^6\alpha) + 6a^2\sin^2\alpha\cos^2\alpha$$

$$= 2a^2[\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha]$$

$$= 2a^2[\sin^6\alpha + (1 - \sin^2\alpha)^3 + 3\sin^2\alpha(1 - \sin^2\alpha)]$$

$$= 2a^2[\cancel{\sin^6\alpha} + 1 - \cancel{\sin^6\alpha} - \cancel{3\sin^2\alpha} + \cancel{3\sin^4\alpha} + \cancel{3\sin^2\alpha} - \cancel{3\sin^4\alpha}]$$

$$= 2a^2$$

Hence $\Rightarrow OP^2 + OQ^2 + 3OT^2 = 2a^2 = \text{constant}$

$\because$ Hint: $(A - B)^3 = A^3 - B^3 - 3A^2B + 3AB^2$

$$(1 - \sin^2\alpha)^3 = 1^3 - (\sin^2\alpha)^3 - 3\cdot 1^2\cdot\sin^2\alpha + 3\cdot 1\cdot(\sin^2\alpha)^2$$

$$(1 - \sin^2\alpha)^3 = 1 - \sin^6\alpha - 3\cdot 1\cdot\sin^2\alpha + 3\cdot 1\cdot\sin^4\alpha$$

$$\cos^6\alpha = (\cos^2\alpha)^3 = (1 - \sin^2\alpha)^3$$

$$\therefore \boxed{\cos^6\alpha = 1 - \sin^6\alpha - 3\sin^2\alpha + 3\sin^4\alpha}$$