This is easily seen:

Let $A_k = a_k + i b_k$, then $\sum_i \sum_k \kappa_{ik} A_i^* A_k$

$$= \sum_i \sum_k \kappa_{ik} (a_i - i b_i)(a_k + i b_k)$$

$$= \sum_i \sum_k \kappa_{ik} a_i a_k + \sum_i \sum_k \kappa_{ik} b_i b_k + i \sum_i \sum_k \kappa_{ik} a_i b_k - i \sum_i \sum_k \kappa_{ik} b_i a_k .$$

The last two terms cancel out because of symmetric κ_{ik}'s. Thus the numerator is a sum of two positive definite terms. Similarly we can check for the denominator. So, we conclude that ω^2 is real and positive.

once the eigenfrequencies ω_α's are known, it is easy to determine the A_k's from the equations

$$\sum_k (-\omega^2 m_{ik} + \kappa_{ik}) A_k = 0$$

If all the roots of the characteristic equation are different, i.e., all the ω_α's are different, then the A_k's are proportional to the minors of the determinant of Eq. (8) with $\omega = \omega_\alpha$. Let these minors be $\Delta_{k\alpha}$. Thus,

$$A_k = C_\alpha \Delta_{k\alpha}, \quad k = 1, 2, \cdots, s \qquad (9)$$

where C_α's are arbitrary complex constants. Further, from eqn (7) and (8), a particular solution obtained is:

$$x_k = C_\alpha \Delta_{k\alpha} \exp(i\omega_\alpha t).$$

The general solution is a sum of the s particular solutions. Choosing the real parts only, we write

$$x_k(t) = \mathrm{Re}\left[\sum_{\alpha=1}^{s} C_\alpha \Delta_{k\alpha} \exp(i\omega_\alpha t)\right] = \sum_\alpha \Delta_{k\alpha}\theta_\alpha, \quad \text{---(10a)}$$

$$\theta_\alpha = \mathrm{Re}\left[C_\alpha \exp(i\omega_\alpha t)\right] \quad \text{---(10b)}$$

Thus, we see that the time variation of each coordinate of the system is a superposition of s simple periodic oscillations $\theta_1, \theta_2, \ldots, \theta_s$, with arbitrary phases and amplitudes but with definite frequencies.

We have s equations of the type Eq. (10a) for s unknown θ's. Conversely we can write s equations for each of the θ's in terms of the co-ordinates $x_1, x_2 \ldots, x_s$. Thus, the set of quantities $\theta_1, \theta_2, \theta_s$ may be considered as a new set of generalized coordinates. From Eq. 10b,

$$\theta_\alpha = \mathrm{Re}\left[C_\alpha \exp(i\omega_\alpha t)\right],$$

so that $$\ddot{\theta}_\alpha = -\omega_\alpha^2 \theta_\alpha.$$

Thus, the equations satisfied by these new coordinates are:

$$\ddot{\theta}_\alpha + \omega_\alpha^2 \theta_\alpha = 0, \quad \alpha = 1, 2, \ldots, s \quad \text{---(11)}$$

These are much simpler than the equations satisfied by the x_k's. θ's are known as the 'normal coordinates' of the mechanical system and they execute simple periodic motion called 'normal oscillations' of the system. We normalize the $\Delta_{k\alpha}$'s in such a manner that

$$\sum_{i,k} m_{ik} \Delta_{ik} \Delta_{k\alpha} = 1 \quad \text{---(12)}$$

This equation may be called the normalization condition on $\Delta_{k\alpha}$. The generalization of Eq. (12) to the case with different indices on Δ's is given by

$$\sum_{i,k} m_{ik} \Delta_{i\alpha} \Delta_{k\beta} = \delta_{\alpha\beta} \qquad \text{——— (12a)}$$

Equation (12a) is called the orthonormalization condition. From Eqs. (8) and (12), we obtain the kinetic and potential energies in terms of the normal coordinates:

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k = \frac{1}{2} \sum_{i,k} \sum_{\alpha,\beta} m_{ik} \Delta_{i\alpha} \dot{\theta}_\alpha \Delta_{k\beta} \dot{\theta}_\beta$$

$$= \frac{1}{2} \sum_{\alpha,\beta} \delta_{\alpha\beta} \dot{\theta}_\alpha \dot{\theta}_\beta = \frac{1}{2} \sum_\alpha \dot{\theta}_\alpha^2 ,$$

$$U = \frac{1}{2} \sum_{i,k} \kappa_{ik} x_i x_k = \frac{1}{2} \sum_{i,k} \sum_{\alpha,\beta} \kappa_{ik} \Delta_{i\alpha} \theta_\alpha \Delta_{k\beta} \theta_\beta$$

$$= \frac{1}{2} \sum_{i,k} \sum_{\alpha,\beta} \omega_\alpha^2 m_{ik} \Delta_{i\alpha} \Delta_{k\beta} \theta_\alpha \theta_\beta = \frac{1}{2} \sum_{\alpha,\beta} \omega_\alpha^2 \delta_{\alpha\beta} \theta_\alpha \theta_\beta$$

$$= \frac{1}{2} \sum_\alpha \omega_\alpha^2 \theta_\alpha^2$$

Thus the Lagrangian of the system can be written as

$$L = T - U = \frac{1}{2} \sum_\alpha \left[\dot{\theta}_\alpha^2 - \omega_\alpha^2 \theta_\alpha^2 \right] \qquad \text{— (13)}$$

Note that with the usual condition

$$\sum_{i,k} \Delta_{i\alpha} \Delta_{k\beta} = \delta_{\alpha\beta} ,$$

we would have obtained the result

$$L = \frac{1}{2} \sum_\alpha m_\alpha \left[\dot{\theta}_\alpha^2 - \omega_\alpha^2 \theta_\alpha^2 \right] \qquad \text{— (14)}$$

With m_α's as positive constants. The transformation

Eq. (109) reduces both T and U to a sum of squares or brings them both on the principal axis. To sum up we note the following salient features of normal coordinates:

(i) Any undamped oscillating system with s degrees of freedom and with s distinct roots of characteristic equation may be described in terms of s normal coordinates $\theta_1, \ldots, \theta_s$. Each normal coordinate θ_r has an eigenfrequency ω_r associated with it.

(ii) The potential energy of the system is a sum of squares of the normal coordinates with all the terms having constant coefficients.

(iii) The kinetic energy of the system is a sum of squares of the time derivatives of the normal coordinates, the coefficients of all the terms being constants.

(iv) Now, the s equations of motion are ordinary linear second-order differential equations with constant coefficients.

(v) The normal coordinates are all independent; no energy is spontaneously transferred to and fro among them.

This treatment needs some modifications if, some roots of the characteristic equation, coincide. The general form of the integral of the equations of motion remains unchanged, but now the coefficients Δ_{kr} corresponding to multiple roots are no longer the minors of the determinant.

~~Illustrate the theory elevated~~

X

Unit 4:- Small Oscillations: Formulation of the problem, the eigenvalue eqn and the principal axis transformation, frequencies of free vibrations and normal conditions, forced vibrations and the effect of dissipative forces. Resonance and beats.