Simpson's 1/3 - Rule:-

This rule is obtained by putting $n=2$ in general formula. i.e by replacing the curve by $n/2$ arcs of second-degree polynomials or paraboles.

General formula:-

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^3}{24}\Delta^3 y_0 + \cdots\right]$$

Putting $n=2$ in this formula, all differences higher than second order will become zero. We have then.

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(\Delta y_1 - \Delta y_0)\right]$$

$$= 2h\left[y_0 + y_1 - y_0 + \frac{1}{6}(y_2 - y_1 - y_1 + y_0)\right]$$

$$= 2h\left[\frac{y_0}{6} + \frac{4y_1}{6} + \frac{y_2}{6}\right]$$

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly, $\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$

and finally $\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$

Summing up we obtain:-

$$\int_{x_0}^{x_n} = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + y_n + 2(y_2 + y_4 + y_6 + \cdots y_{n-2})\right]$$

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Hence,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + y_n\right]$$

Which is known as simpson's $\frac{1}{3}$-rule or simpson's rule. Thus, this rule requires the division of the whole range into the even number of subintervals of width h.

Following the method can be shown that the error in simpson's rule is given by.

$$\int_a^b y\,dx \doteq \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + y_n\right]$$

$$= \frac{b-a}{180} h^4 y^{iv}(\bar{x})$$

where $y^{iv}(x)$ is the Largest value of the fourth derivatives.

Simpson's $\frac{3}{8}$-Rule :-

Setting $n=3$ in the general formula, we observe that all the differences higher than the third will become zero; Then-

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly $\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$

and so on.

$$\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Summing, then.

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

This rule, called Simpson's (3/8)-rule. and error of this formula being $\left(-\frac{3}{80}\right) h^5 y^{iv}(\bar{x})$.

Exp. A solid of revolution is formed by the rotating about the x-axis the area between the x-axis, the lines $x=0$ and $x=1$ and a curve through the points with the following co-ordinates:-

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

estimate the volume of the solid formed.

Solution If V is the volume of the solid formed then $V = \pi \int_0^1 y^2\,dx$ and $h = 0.25$

	x	y	y^2	
x_0	0.0	1.0000	1.0000	y_0
x_1	0.25	0.9896	0.9793	y_1
x_2	0.50	0.9589	0.9195	y_2
x_3	0.75	0.9089	0.8261	y_2
x_4	1.00	0.8415	0.7081	y_4

Using Simpson's $\frac{1}{3}$ rule, we get

$$V = \pi \int_0^1 y^2 dx = \frac{\pi h}{3} \int_0^1 y^2 dx$$

$$V = \frac{\pi (0.25)}{3} \left[y_0 + 4(y_1 + y_3) + 2(y_2 + y_4) \right]$$

$$= \frac{3.14 \times 0.25}{3} \left[1.0000 + 4(0.9793 + 0.8261) + 2(0.9195) + 0.7081 \right]$$

$$= 0.2617 (1.0000 + 7.2216 + 2.5477)$$

$$= 0.2617 \times 10.7687$$

$$V \simeq 2.818$$