only interested in real roots of the above equation. ω_k's determined from this equation are known as **characteristic feequencies** or **eigenfrequencioes.**
From physical arguments it is clear that for real physical situations, the roots are real and positive. This is because the existence of an imaginary part in ω would mean time dependence of q_k and $\dot{q}_k$ such that the total energy would not be conserved in time and such solutions are unacceptable.
We can arrive at the same conclusion mathematically as well. Multiplying (6) with A_k^* and summing over k we get

$$\sum_{i,k}(V_{ik} - \omega^2 t_{ik})A_k^* A_i = 0$$

so that

$$\omega^2 = \frac{\sum_{i,k} V_{ik} A_k^* A_i}{\sum_{i,k} t_{ik} A_k^* A_i}$$

Both the numerator and the denominator are real because $V_{ik} = V_{ki}$ and $t_{ik} = t_{ki}$. It is seen that the terms are positive as well because expressing $A_i = a_i + ib_i$, we have

$$\begin{aligned}\sum_{i,k} t_{ik} A_i^* A_k &= \sum_{i,k} t_{ik}(a_i - ib_i)(a_k + ib_k)\\ &= \sum_{i,k} t_{ik}(a_i a_k + b_i b_k)\end{aligned}$$

where the imaginary terms cancel because of symmetry of t_{ik}. Thus we have been able to express $\sum_{i,k} t_{ik} A_i^* A_k$ as a sum of two positive semi-definite terms ($\sum t_{ik} a_i a_k = a^T t \dot{a}$ is positive definite).

2.2 Matrix Formulation

Let us rewrite (6) (using symmetry properties of V and T) as

$$\sum_i (V_{ki} - \lambda t_{ki})A_i = 0$$

where $\lambda = \omega^2$. Let us define a column vector

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_N \end{pmatrix}$$

The matrices V and t are given by

$$V = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1N} \\ V_{21} & V_{22} & \cdots & V_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ V_{N1} & V_{N2} & \cdots & V_{NN} \end{pmatrix} \qquad T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & t_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix}$$

Using these, the equation (6) can be expressed as a matrix equation

$$VA = \omega^2 TA \tag{7}$$

Note that this equation is not in the form of an eigenvalue equation as VA is not equal to a constant times A but a constant times TA. (If T is invertible, one can get an eigenvalue equation $T^{-1}VA = \omega^2 IA$.

Since we have N homogeneous equations, we have N modes, i.e. N solutions for ω^2. Let us denote the k-th mode frequency by $\omega_k^2 = \lambda_k$. Let the vector A corresponding to this mode be written as

$$A_k = \begin{pmatrix} A_{k1} \\ A_{k2} \\ \dots \\ A_{kN} \end{pmatrix}$$

We then have

$$VA_k = \lambda_k TA_k \tag{8}$$

Taking conjugate of this equation and changing the index k to i, we get

$$\tilde{A}_i V = \lambda_i \tilde{A}_i T \tag{9}$$

where we have used $\tilde{A}$ to denote the transpose of the matrix A.. From (8) we get by multiplying with $\tilde{A}_k$

$$\lambda_k = \frac{\tilde{A}_k VA_k}{\tilde{A}_k TA_k} \tag{10}$$

From (8) and (9) it follows that

$$\tilde{A}_i VA_k = \lambda_k \tilde{A}_i TA_k$$
$$\tilde{A}_i VA_k = \lambda_i \tilde{A}_i TA_k$$

so that

$$(\lambda_k - \lambda_i)\tilde{A}_i TA_k = 0$$

Thus, if the eigenvalues are non-degenerate, i.e. if $\lambda_i \neq \lambda_k$, we get the *orthogonality condition*

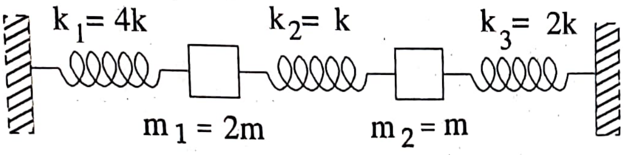
$$\tilde{A}_i TA_k = 0 \tag{11}$$

Note that this is different from the orthogonality condition on eigen vectors for a regular eigenvalue equation. Since (7) does not uniquely determine A, we define **normalization** condition as

$$\tilde{A}_i TA_i = 1 \tag{12}$$

Example 1:

Consider two masses $m_1 = 2m$ and $m_2 = m$ connected by three springs, as shown.



We know that a single mass spring system with mass m and spring constant k has a natural frequency of oscillation $\sqrt{k/m}$. Let us consider the system in the figure. Let the generalised coordinates be displacement of the masses from their equilibrium positions, the mass m_1 being displaced by x_1 while the mass m_2 by an amount x_2. The central spring is then compressed or stretched by an amount $x_2 - x_1$. Let us attempt to solve the problem using the force method. The equations of motion for m_1 and m_2 are

$$m_1\ddot{x}_1 = -4kx_1 - k(x_1 - x_2) \tag{13a}$$

$$m_2\ddot{x}_2 = -2kx_2 - k(x_2 - x_1) \tag{13b}$$

The sign of the last term is fixed by taking x_2 to be large positive so that the force on m_1 is in the positive direction, as it ought to be.
We will attempt to solve this pair of coupled equations (13a) and (13b) by using a bit of guesswork and a bit of luck. This will not work except in cases which show sufficient symmetry. The idea is to find a linear combination of x_1 and x_2 so that the coupled equations become uncoupled. Let us rewrite (13a) and (13b) as

$$2m\ddot{x}_1 = -5kx_1 + kx_2 \tag{14a}$$

$$m\ddot{x}_2 = kx_1 - 3kx_2 \tag{14b}$$

It can be easily seen that

$$m(\ddot{x}_1 - \ddot{x}_2) = -\frac{7}{2}k(x_1 - x_2) \tag{15a}$$

$$m(2\ddot{x}_2 + \ddot{x}_1) = -2k(2x_1 + x_2) \tag{15b}$$

Thus, if we define new **normal coordinates**

$$\begin{aligned} y_1 &= x_1 - x_2 \\ y_2 &= 2x_1 + x_2 \end{aligned} \tag{16}$$

the equations for y_1 and y_2 are uncoupled,

$$\begin{aligned} m\ddot{y}_1 &= -\frac{7}{2}ky_1 \\ m\ddot{y}_2 &= -2ky_2 \end{aligned} \tag{17}$$

The normal coordinates oscillate with independent frequencies $\sqrt{2k/m}$ and $\sqrt{7k/2m}$, which are known as the **normal modes**. These new generalised coordinates execute simple periodic oscillations known as **normal oscillations**. The new normal coordinates satisfy

$$\ddot{y}_\alpha + \omega_\alpha^2 y_\alpha = 0 \tag{18}$$

With this intuitive background, let us look at the problem more formally.
The Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2 - \frac{1}{2}k_3x_2^2 \tag{19}$$

Define t and V matrices

$$t = \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{20}$$

$$V = \frac{\partial^2 V}{\partial x_i \partial x_j} = \begin{pmatrix} 5k & -k \\ -k & 3k \end{pmatrix} \tag{21}$$

Condition for existence of non-trivial solution is

$$\det(V_{ik} - \omega^2 t_{ik}) = 0$$

which gives

$$\begin{vmatrix} 5k - 2m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0$$

which has the solution $\omega^2 = 7k/2m$ and $2k/m$. The normal modes are now found from (6)

$$\omega^2 TA = VA$$

Writing $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, we get, for $\omega^2 = \dfrac{7k}{2m}$

$$\frac{7k}{2m}\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 5k & -k \\ -k & 3k \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

For this **normal mode** we have $A_2 = -2A_1$ so that (normalizing)

$$A = \frac{1}{\sqrt{6m}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\omega^2 = \dfrac{2k}{m}$, a similar calculation gives

$$A = \frac{1}{\sqrt{3m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution can then be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_- t + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin\omega_- t + C\begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos\omega_+ t + D\begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin\omega_+ t$$

We can fix the constants A, B, C and D by using initial conditions. Suppose we we pull the two masses aside and release,

$$x_1(0) = x_{10}, x_2(0) = x_{20}, \dot{x}_1(0) = \dot{x}_2(0) = 0$$

we then have

$$\begin{aligned} A + C &= x_{10} \\ A - 2C &= x_{20} \\ B\omega_- + D\omega_+ &= 0 \\ B\omega_- - 2D\omega_+ &= 0 \end{aligned}$$

which gives

$$\begin{aligned} A &= \frac{2}{3}x_{10} + \frac{1}{3}x_{20} \\ B = D &= 0 \\ C &= \frac{1}{3}x_{10} - \frac{1}{3}x_{20} \end{aligned}$$

so that

$$x_1 = \left(\frac{2}{3}x_{10} + \frac{1}{3}x_{20}\right)\cos\omega_- t + \left(\frac{1}{3}x_{10} - \frac{1}{3}x_{20}\right)\cos\omega_+ t \quad (22)$$

$$x_2 = \left(\frac{2}{3}x_{10} + \frac{1}{3}x_{20}\right)\cos\omega_- t - \left(\frac{2}{3}x_{10} - \frac{2}{3}x_{20}\right)\cos\omega_+ t \quad (23)$$

It can be checked that,

$$\begin{aligned} x_1 - x_2 &= (x_{10} - x_{20})\cos\omega_+ t \\ 2x_1 + x_2 &= (2x_{10} + x_{20})\cos\omega_- t \end{aligned}$$

as expected.

Example 2:

As a second example, consider a pair of identical pendulums consisting of a pair of mass-less rigid rods of length l each at the end of each of which a mass m is attached. The mid-points of the rods are connected by a spring of force constant k. In the absence of coupling, each oscillator has a frequency $\omega = \sqrt{\frac{g}{l}}$. As before, let us try to solve the problem by force method.