Reduction to triangular forms

Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 6 \\ 0 & 0 & 4 & 6 \end{bmatrix}$

To find det(A)?

How we will reduce A until it is triangular form by row elementary operation

$R_1 \to R_1 - 2R_2$ & $R_1 \to R_2$

$\to \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 6 \\ 0 & 0 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 1 & 1 & 3 & 6 \\ 0 & 0 & 4 & 6 \end{bmatrix}$

$R_3 \to R_3 - R_1$; $R_3 \to R_3 - R_2$

$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 4 & 6 \end{bmatrix}$

$R_3 \to \frac{1}{2} R_4$

$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B \Rightarrow$ This is in triangular form

$\det(B) = 1 \cdot 1 \cdot 2 \cdot 1 = \frac{2}{2}$

Theorem - If $\dim V = n$ and if $T \in A(V)$ has all its characteristic roots in F. Then T satisfies a polynomial of degree n over F.

Proof:- Let $\lambda_1, \lambda_2 --- \lambda_n$ be the characteristic roots of F. Since T be all its characteristic roots in F so there is a basis $(v_1, v_2 \cdots v_n)$ of V such that

$$T(v_1) = \lambda_1 v_1$$

$$T(v_2) = a_{21}\lambda_1 + \lambda_2 v_2$$

$$T(v_3) = a_{31}\lambda_1 + a_{32}\lambda_2 + \lambda_3 v_3$$

$$- \quad --- \quad -$$

$$T(v_n) = a_{n1}\lambda_1 + a_{n2}\lambda_2 + \cdots\cdots + \lambda_n v_n$$

How. these Equations can be written as

$$T(v_1) - \lambda_1 v_1 = 0 \Rightarrow (T - \lambda_1 I) v_1 = 0 \quad \text{—(i)}$$

$$(T - \lambda_2 I) v_2 = a_{21} v_1 \quad \text{—— (ii)}$$

$$(T - \lambda_3 I) v_3 = a_{31} v_1 + a_{32} v_2 \quad \text{—— (iii)}$$

$$- \quad --- \quad -$$

$$- \quad - \quad --$$

$$(T - \lambda_n I) v_n = a_{n1} v_1 + a_{n2} v_2 + \cdots + a_{n-1} v_{n-1} \quad \text{— (n)}$$

from (i) & (ii) from $v_1 = \frac{1}{a_{21}}(T - \lambda_2 I)v_2$ put in (i)

$$(T - \lambda_1 I)(T - \lambda_2 I)\frac{1}{a_{21}} v_2 = 0$$

$$(T - \lambda_1 I)(T - \lambda_2 I) v_2$$

Now $(T-\lambda_2 I)(T-\lambda_1 I)v_2 = (T-\lambda_1 I)(T-\lambda_2 I)v_2$

$[\because (T-\lambda_2 I)(T-\lambda_1 I) = (T-\lambda_1 I)(T-\lambda_2 I)]$

$= (T-\lambda_1 I)\, a_{21} v_1$ from (ii)

$= a_{21}(T-\lambda_1 I)v_1$

$(T-\lambda_2 I)(T-\lambda_1 I)v_2 = a_{21}\cdot 0 = 0$ from (i)

$(T-\lambda_2 I)(T-\lambda_1 I)v_2 = 0$

and

$(T-\lambda_3 I)(T-\lambda_2 I)(T-\lambda_1 I).v_3 = (T-\lambda_2 I)(T-\lambda_1 I)(T-\lambda_3 I)v_3$

$= (T-\lambda_2 I)(T-\lambda_1 I)[a_{31}v_1 + a_{32}v_2]$

$= (T-\lambda_2 I)(T-\lambda_1 I)a_{31}v_1 + (T-\lambda_2 I)(T-\lambda_1 I)a_{32}v_2$

$= (T-\lambda_2 I).a_{31}(T-\lambda_1 I)v_1 + (T-\lambda_1 I)a_{32}(T-\lambda_2 I)v_2$

$= a_{31}(T-\lambda_2 I)(T-\lambda_1 I)v_1 + a_{32}(T-\lambda_2 I)(T-\lambda_1 I)v_2$

$= 0 + 0$

$(T-\lambda_3 I)(T-\lambda_2 I)(T-\lambda_1 I)v_3 = 0$

Similarly continuing in this way we get

$(T-\lambda_n I)(T-\lambda_{n-1} I)\cdots\cdots(T-\lambda_2 I)(T-\lambda_1 I)v_n = 0$

Let $S = (T-\lambda_n I)(T-\lambda_{n-1} I)\cdots(T-\lambda_1 I)$, where

satisfies $S(v_1) = 0 = S(v_2) = 0 = \cdots S(v_n)$

Then S annihilates a basis of V thus S annihilates all of V. Therefore $S=0$ which implies that $(T-\lambda_n I)(T-\lambda_{n-1} I)\cdots(T-\lambda_1 I) = 0$

Hence T satisfies the polynomial.