To find the matrix of transformation from

$|A-\lambda I|X=0$

$$A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$$ $\rightarrow$ This matrix A from quadratic form $q=3x^2+5y^2+3z^2+2zx+2yz-2xy$

Then the characteristic Equation $|A-\lambda I|=0$ then $\lambda=2,3,6$ are eigen values.

Now

$$[A-\lambda I]X=0 \Rightarrow \begin{bmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

We obtain the Equations.

$$\left.\begin{aligned}(3-\lambda)x-y+z&=0\\ -x+(5-\lambda)y-z&=0\\ x-y+(3-\lambda)z&=0\end{aligned}\right\}\quad -(1)$$

To find eigen vectors from corresponding eigen values from Eqn(1). Then

(i) For $\lambda=2$ putting in Eqn(1)

$x-y+z=0$ — (i)

$-x+2y-z=0$ — (ii)

$x-y+z=0$ (iii)

from (i) + (ii)

$y=0$

put $y=0$ in (i) then

$x+z=0 \Rightarrow x=-z$

Take $z=1$ then $x=-1$

∴ Whence $x=-1$, $y=0$ & $z=1$

∴ Eigen vector $X_1=[-1,0,1]$ and its normalized $\|X_1\|=\sqrt{1+0+1}=\sqrt{2}$

form X_{n} $[-1/\sqrt{2},\ 0,\ 1/\sqrt{2}]$

(ii) Similarly, for $\lambda=3$ putting in Eqn(1) for Eigen vector X_2

we get Eqn from (1) $0\cdot x-y+z=0$, $\Rightarrow y=z$

$-x+2y-z=0$

$x-y+0\cdot z=0 \Rightarrow x=y$

$x=y=z$ then $x=1$, $y=1$ & $z=1$

Eigen vector X_2 (for $\lambda = 3$)

$X_2 = [1, 1, 1]$ and its normalized form

$\|X_2\| = \sqrt{1+1+1} = \sqrt{3}$

$nor\,X_2 = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]$

(iii) for $\lambda = 6$ putting in Eqn (1), we get Eqn for Eigen vectors X_3.

$-3x - y + z = 0$ — (i)

$-x - y - z = 0$ — (ii)

$x - y - 3z = 0$ — (iii)

from (i) + (ii)

$-4x - 2y = 0 \Rightarrow 2x = -y$ — (iv)

from (ii) + (iii)

$-2y - 4z = 0 \Rightarrow -y = 2z$ — (v)

from (iv) & (v)

$\Rightarrow 2x = 2z \Rightarrow x = z$ take $z = 1$

$x = 1$, $y = -2$ & $z = 1$

Eigen vector for $X_3 = [1, -2, 1]$ and its normalized form

$\|X_3\| = \sqrt{1+4+1}$

$\|X_3\| = \sqrt{6}$

$nor\,X_3 = \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$

Hence the matrix of transformation

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$\because B = P^T A P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & \frac{1}{\sqrt{3}} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \underbrace{-3/\sqrt{2} + 0 + 1/\sqrt{2}}_{-2/\sqrt{2}}, & \frac{3}{\sqrt{3}}, & 6/\sqrt{6} \\ 0, & 3/\sqrt{3}, & -12/\sqrt{6} \\ 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{bmatrix}$$

$$B = \begin{bmatrix} 1+0+1 & -\frac{3}{\sqrt{6}} + 0 + \frac{3}{\sqrt{6}} & -\frac{6}{\sqrt{12}} + 0\,\frac{6}{\sqrt{12}} \\ -\frac{2}{\sqrt{6}} + 0\,\frac{2}{\sqrt{6}} & 1+1+1 & \frac{6}{\sqrt{18}} - \frac{12}{\sqrt{18}} + \frac{6}{\sqrt{18}} \\ -\frac{2}{\sqrt{12}} + 0\,\frac{2}{\sqrt{12}} & \frac{3}{\sqrt{18}} - \frac{6}{\sqrt{18}} + \frac{3}{\sqrt{18}} & \frac{6}{6} + \frac{24}{6} + \frac{6}{6} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Exp. Reduce quadratic form $q = f(x)$ into Canonical form if $q = 3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by Linear Transformation.

Sol. Given quadratic form

$$q = 3x^2 + 0 \cdot y^2 + 3z^2 + 4xy + 8xz + 8yz \quad -①$$

$$\Rightarrow a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

$\because$ $a_{11} = 3$, $a_{22} = 0$, $a_{33} = 3$, $2a_{12} = 4 \Rightarrow a_{12} = 2$,

$2a_{13} = 8 \Rightarrow a_{13} = 4$, $2a_{23} = 8 \Rightarrow a_{23} = 4$

then symmetric matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

where $a_{ij} = a_{ji}$

Where $q = X'AX$ where $X' = [x, y, z]$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$$

Reduce matrix A into diagonal matrix and canonical form by L.T. using Elementry Row/Column operations.

We write $A = I^T A I$

$A \rightarrow$ given

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}}_{\text{R \& C operations one by one}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{only Row operation}} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{only column operation}}$$

* Symmetric matrix $A = I^T A I$ is reduced to diagonal matrix on applying Row and column operations on A but Row operations on I^T and corresponding column operations on I to the form.

$$\begin{array}{c} A \\ \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} \\ \uparrow \end{array} = \begin{array}{c} I^T \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \uparrow \\ R\text{-Op.} \end{array} A \begin{array}{c} I \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \uparrow \\ C\text{-Op.} \end{array}$$

Applying R-C-Operations

$R_2 \to R_2 - \frac{2}{3}R_1$ & $R_3 \to R_3 - \frac{4}{3}R_1$

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -4/3 & 4/3 \\ 0 & 4/3 & -7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying corresponding column operation

$C_2 \to C_2 - \frac{2}{3}C_1$, $C_3 \to C_3 - \frac{4}{3}C_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 4/3 \\ 0 & 4/3 & -7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2/3 & -4/3 \\ \cancel{-2/3} & 1 & 0 \\ \cancel{-4/3} & 0 & 1 \end{bmatrix}$$

Again applying Row operation

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2/3 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying corres. column operation

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & -2/3 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2/3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$D(3, -4/3, -1) = P^T A P$$

$\Downarrow$

B

Matrix B is called diagonal matrix.

$\therefore$ The canonical form of the given quadratic form $q = X^T A X$ is $Y' B Y$. Then

$$Y'BY = [x, y, z] \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $Y' = [x, y, z]$

$$Y'BY = 3x^2 - \frac{4}{3}y^2 - z^2$$

(i) Rank of B, $r = 3$, (No. of non-zero eigen values or terms)

(ii) Index $p = 1$ (No. of positive terms in canonical form or positive eigen values)

(iii) Signature $= 1 - 2 = -1$ (Diff b/w (+ve) and (-ve) eigenvalues or terms in C.F.)

(iv) Nature $\to$ Semi-positive and Negative