Quotient Space: Let W be subspace of V over F. Let $u \in V$ be arbitrary vector

Then $u+W = \{u+w : w \in W\}$ is called a coset of W in Vector space V.

The collection of all cosets of W in V denoted by V/W as

$$\frac{V}{W} = \{u+W : u \in V\}$$

The operations of vector addition and scalar multiplication on V/W as follows

(i) $(u+W) + (v+W) = (u+v)+W \qquad u, v \in V$

(ii) $\alpha(u+W) = \alpha u + W \qquad \alpha \in F$

Then V/W is vector space over field F. This is called quotient of V/W (V by W).

First, we show that compositions/operations are well defined.

Let $u+W = u'+W$ and $v+W = v'+W$

Then $u-u' \in W$ & $v-v' \in W$

$\Rightarrow (u-u') + (v-v') \in W$, since W is subspace of V.

$(u+v) - (u'+v') \in W$

$\Rightarrow (u+v)+W = (u'+v')+W$

Again $u+W = u'+W \Rightarrow (u-u')+W \Rightarrow u-u' \in W$

for $\alpha \in F$ then

$\alpha(u-u') \in W = \alpha u - \alpha u' \in W$

$= \alpha u + W = \alpha u' + W$

Hence the addition and scalar multiplication as given (i) and (ii) are well defined for V/W.

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Invariant Subspaces:- Let $T: V \to V$ be a linear transformation.

Then a subspace W of V is invariant under T if $T(W) \subseteq W$ i.e. if $u \in W$, then $T(u) \in W$.

Theorem:- If W is a subspace invariant under $T \in A(V)$, then T induces/produces a linear transformation $\bar{T}$ on V/W (Quotient space) defined by $\bar{T}(u+W) = T(u)+W$. Moreover, if T satisfies the polynomial $q(x) = F[x]$, then so is $\bar{T}$. Thus the minimal polynomial of $\bar{T}$ divides the minimal polynomial of T.

Proof:- Firstly, we show that $\bar{T}$ is well defined.

Let $u+W$ and $v+W$ be any elements of V/W (Quotient space, read as V by W).

If $u+W = v+W$. then $u-v \in W$

Since W is T-invariant, then

$$T(u-v) = T(u) - T(v) \in W \quad \text{by definition}$$

So $T(u)+W = T(v)+W$

$$\bar{T}(u+W) = \bar{T}(v+W)$$

Thus $\bar{T}$ is we deffined.

Now, we show that $\bar{T}$ is linear. for this

$$\bar{T}[(u+W)+(v+W)] = \bar{T}[(u+v)+W]$$

$= T(u+v)+W$ from the condition

$\because T$ is linear

$$= T(u)+T(v)+W$$

$$= T(u)+W + T(u)+W$$

$\therefore \;\; \breve{T}[(u+W)+(v+W)] = (T(u)+W)+(T(v)+W)$

$= \breve{T}(u+W)+\breve{T}(v+W)$

Also $\;\; \breve{T}[\alpha(u+W)] = \breve{T}[\alpha u+W]$

$= T(\alpha u)+W$

$= \alpha T(u)+W$

$= \alpha[T(u)+W]$

$= \alpha \breve{T}(u+W)$

Thus $\breve{T}$ is linear.

If $u+W \in V/W$, then

$\breve{T}^2(u+W) = T^2(u)+W$

$= T(T(u)+W$

$= \breve{T}[T(u)+W]$

$= \breve{T}[\breve{T}(u+W)]$

$= (\breve{T})^2(u+W)$

$\therefore \; \breve{T}^2 = (\breve{T})^2$

$\Rightarrow \; (\breve{T^2}) = (\breve{T})^2$

Similarly, $(\breve{T^n}) = (\breve{T})^n$ for any $n \geq 0$.

Now for any polynomial $q(x) = F\{x\}$ given by $q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$\Rightarrow q(\breve{T})(u+W) = q(T)(u)+W$

$= a_n T^n(u) + a_{n-1}T^{n-1}(u) + \cdots a_0 I(u) + W$

$$q(\tilde{T})(u+W) = q(T)(u)+W = \sum a_i T^i(u)+W$$
$$= \sum a_i (T^i(u)+W)$$
$$= \sum a_i \tilde{T}^i(u+W)$$
$$= \sum a_i (\tilde{T})^i (u+W)$$
$$= q(\tilde{T})(u+W)$$

$\therefore\ q(\tilde{T}) = q(\bar{T})$.

According if T is a root if $q(x)=0$ then

$q(\bar{T}) = \tilde{0} = W = q(\bar{T})$

Thus $\tilde{T}$ is also a root of $q(x)=0$

Let $p_1(x)$ be the minimal polynomial over F satified by $\tilde{T}$.

$\Rightarrow$ If $q(\bar{T})=0$ for $q(x)=F[x]$, then $p_1(x)/q(x)$.

If $p(x)$ is the minimal polynomial of T over F then $p(T)=0 \Rightarrow p(\tilde{T})=0$

Hence $p_1(x)/q(x)$