Quadratic Forms- A quadratic form is a homogeneous polynomial of degree two in a number of variables. Exp. $ax^2+2xy+by$ is a quadratic form in the variables x & y.

Quadratic forms are homogeneous quadratic polynomials (i.e degree of two) in n variables. In cases of one, two and three variables they are called unary, binary and ternary respectively and have the following form.

(i) $q(x) = ax^2$ (Unary

(ii) $q(x,y) = ax^2 + bxy + cy^2$ (Binary)

(iii) $q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$ (Ternary)

where $a, b, \cdots f$ are the coefficients.

Real Quadratic form:-

Definition A mapping (function) $q: V \to F$ is called a quadratic form if $q(v) = f(v,v)$ for some bilinear form f on V.

* A quadratic form is a polynomial $q(X) = X^T A X$ where $X^T = [x_1, x_2 \cdots x_n]$ and A is symmetric matrix. That is $q(X) = [x_1, x_2 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ - & - & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i \cdot x_j$$

$$q(X) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots a_{nn}x_n^2 + 2\sum_{i<j} a_{ij} x_i \cdot x_j \quad \begin{matrix} i = 1 \text{ to } n \\ j = 1 \text{ to } n \end{matrix}$$

clearly, $q(X)$ is a polynomial in homogeneous second degree.

⟹ 1- If $q(x) = x^T A x$ is a real quadratic form, then the rank of q is defined as the rank of the matrix A.

i.e $\text{rank}(q) = \text{rank}(A) = \rho(A)$

⟹ 2- Let q be a real quadratic form and f be a bilinear form, then the signature of f and of q are defined by $\text{sig.}(f) = \text{sig.}(q) = P - N$,
where p is the number of positive entries and N the negative entries (eigenvalues) in any diagonal representation of f and q.

⟹ 3- If the matrix A is diagonal then the corresponding quadratic form q has diagonal representation as

$$q(x) = x^T A x = a_{11}x_1^2 + a_{22}x_2^2 + \cdots a_{nn}x_n^2.$$

So, that the corresponding symmetric matrix A is diagonal

Sylvester's Theorem or Law of Inertia:-

Statement- Let f be a symmetric bilinear form on V over R (real field). Then there is a basis of V in which f is represented by a diagonal matrix, every other diagonal representation has the same number P of positive entries and the same number N of negative entries.

Proof: Let there is a basis $\{u_1, u_2 --- u_n\}$ of V such that $f(u_i, u_j) = 0$ for $i \neq j$

This shows that f is represented by a diagonal matrix with P (positive) and N (negative) entries.

Now assume that $\{v_1, v_2 --- v_n\}$ is another basis of V in which f is represented by a diagonal matrix with P_1 and N_1 respectively positive and negative entries.

We may assume that positive entries in each matrix representation appear first.

Since $\text{rank}(f) = P + N = P_1 + N_1$

Now we prove that only $P = P_1$.

Let U be a subspace of V spanned by the vectors $u_1, u_2 \cdot - \cdot u_p$ and let W be a subspace of V spanned by $v_{p+1}, v_{p+2}, -- v_n$.

Then $f(u, u) > 0$ $\forall$ non-zero $u \in U$ and

$f(u, u) \leq 0$ $\forall$ non-zero $u \in W$.

$\Rightarrow \quad U \cap W = \{0\}$

$\dim U = p$

$\dim W = n - p_1$

Thus $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$= p + n - p_1 - 0$

$\dim(U+W) = p - p_1 + n \quad \text{——(i)}$

But $\dim(U+W) \leq \dim V = n$

$\dim(U+W) \leq n \quad \text{——(ii)}$

from (i) & (ii)

$p - p_1 + n \leq n$

or

$p \leq p_1$.

Similarly, $p_1 \leq p$

therefore $p = p_1$

Hence the theorem.

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