Quotient Space

Quotient space – Let W be a subspace of a vector space $V(F)$. Let $v \in V$ be arbitrary. Then

$$v+W = \{v+w : w \in W\}$$ is called a coset of W in V.

The collection of all cosets of W in V denoted by V/W. Thus

$$V/W = \{v+W : v \in V\}$$

Now, we define the operations of vector addition and scalar multiplication on V/W as follows –

(i) $(u+W)+(v+W) = (u+v)+W$

(ii) $\alpha(u+W) = \alpha u + W$, $\forall\, u, v \in V$ & $\alpha \in F$

Then V/W is vector space over F.

This is vector space is called quotient of V by W.

Firstly, we show that the compositions are well defined.

Let $u+W = u'+W$ and $v+W = v'+W$

Then $u-u' \in W$ & $v-v' \in W$

$\Rightarrow (u-u')+(v-v') \in W$, since W is subspace of V.

$\Rightarrow (u+v)-(u'+v') \in W \Rightarrow (u+v)+W = (u'+v')+W$

Again $u+W = u'+W \Rightarrow (u-u')+W \Rightarrow u-u' \in W$

for $\alpha \in F$, we have

$\alpha(u-u') \in W \Rightarrow \alpha u + W = \alpha u' + W$

Hence the addition and scalar multiplication as given in (i) & (ii) are well defined for V/W.

It is easy to verify that V/W is an abelian group w.r.t. addition composition (1). where

(i) $\bar{0} = 0 + W = W$ is additive identity and

(ii) $-u + W \in V/W$ is the additive inverse of $u + W \in V/W$

Let $u+W, v+W \in V/W$ and $\alpha, \beta \in F$ then by compositions (1) & (2), we have

(I) $(\alpha+\beta)(u+W) = (\alpha+\beta)u + W$

$= \alpha u + \beta u + W$

$= (\alpha u + W) + (\beta v + W)$

$= \alpha(u+W) + \beta(u+W)$

(ii) $\alpha[(u+W) + (v+W)] = \alpha[(u+v) + W]$

$= \alpha(u+v) + W$

$= \alpha u + \alpha v + W$

$= \alpha(u+W) + \alpha(v+W)$

(iii) $(\alpha\beta)(u+W) = (\alpha\beta)u + W$

$= \alpha(\beta u) + W$

$= \alpha[\beta u + W]$

$= \alpha[\beta(u+W)]$

(iv) $1(u+W) = 1u + W = u + W$

Hence V/W is a vector space over F

V/W is called the Quotient space of V by W.

Theorem Let W be a subspace of a finite dimensional vector space $V(F)$. Then

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$

Proof Let $\dim V = n$ and $\dim W = m$ so that $m \leq n$

Let $S_1 = \{x_1, x_2, \ldots x_m\}$ be basis of W. There m vectors are L.I in W are also L.I in V and So they can be extended to form a basis of V.

Let $S_2 = \{x_1, x_2, \ldots x_m, x_{m+1}, \ldots x_n\}$ be basis of V.

Let $S_3 = \{x_{m+1}+W, x_{m+2}+W, \ldots x_n+W\}$

We show that S_3 (which consists $n-m$ vectors) is a basis of V/W.

Let $x+W \in V/W$ be arbitary

Since S_2 is a basis of V, so $x \in V$ can be written as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_m x_m + \alpha_{m+1} x_{m+1} + \ldots \alpha_n x_n$$

$x \in W$

$$\Rightarrow x+W = \{\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_m x_m + \ldots \alpha_n x_n\} + W$$

$$= (\alpha_{m+1} x_{m+1} + \ldots \alpha_n x_n) + (\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_m x_m) + W$$

$$= (\alpha_{m+1} x_{m+1} + \alpha_{m+2} x_{m+2} + \ldots \alpha_n x_n) + W \quad (\because \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_m x_m \in W)$$

$$= \alpha_{m+1}(x_{m+1}+W) + \alpha_{m+2}(x_{m+2}+W) + \ldots$$

$$+ \alpha_n (x_n + W)$$

which is a linear combination of the elements of S_2.

$\therefore \frac{V}{W} = L(S_2)$, Now we show that S_3 is a L.I set

Let $\alpha_{m+1}(x_{m+1}+W)+\alpha_{m+2}(x_{m+2}+W)+\cdots\alpha_n(x_n+W)$
$= \bar{0}$ $\alpha_i \in F$ ——①

$\Rightarrow (\alpha_{m+1}x_{m+1}+\alpha_{m+2}x_{m+2}+\cdots\alpha_n x_n)+W = W$

$\because \bar{0} = W \in V/W$

$\Rightarrow \alpha_{m+1}x_{m+1}+\cdots\alpha_n x_n \in W$

$\Rightarrow \alpha_{m+1}x_{m+1}+\alpha_{m+2}x_{m+2}+\cdots\alpha_n x_n = \beta_1 x_1+\beta_2 x_2+\cdots+\beta_m x_m \in F$

$\because W = L(S_1)$

$\Rightarrow \beta_1 x_1+\beta_2 x_2+\cdots\beta_m x_m+(-\alpha_{m+1}x_{m+1})+(-\alpha_{m+2}x_{m+2})+\cdots\cdots(-\alpha_n x_n) = 0$

$\Rightarrow \beta_1 = \beta_2 \cdots = \beta_m = 0$ and $\alpha_{m+1} = \alpha_{m+2} = \cdots \alpha_n = 0$

Since S_2 is a L.I set.

Using $\alpha_{m+1} = \alpha_{m+2} = \cdots = \alpha_n = 0$ in Eq ①

It follows that S_1 is a L.I set and $L(S_2) = V/W$.

Hence S_3 consisting $n-m$ vectors is a basis of V/W and so

$$\dim V/W = n-m = \dim V - \dim W$$

Exp. Determine $\dim V/W$, where $V = C(R)$ & $W = R(R)$.

Sol. $\dim V = 2$ i.e basis of $V = \{1, i\} \to C(R)$

$\dim W = 1$ i.e basis of $W = \{1\} \to R(R)$

Exp.

$\dim V = 2$

$\dim W = 1$

$\Rightarrow \dim \frac{V}{W} = 2 - 1 = 1$

$\left[\frac{V}{W} = (v+W : v \in W)\right]$