Exp. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$

Solution Given equation

$$y^2 = 4ax \quad \text{—— (i)}$$

Differentiating Eqn (i) w.r.to x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y} = \frac{y^2}{2x \cdot y} = \frac{y}{2x} \quad \text{From (i)}$$

$$\frac{dy}{dx} = \frac{y}{2x} \Rightarrow y' = \frac{y}{2x}$$

or $y - 2xy' = 0$ —— (ii)

For orthogonal trajectories we replacing

$y' = -\frac{1}{y_1}$ in Eqn (ii) then

$$y - 2x\left(-\frac{1}{y_1}\right) = 0$$

$$y \cdot y' + 2x = 0$$

$$y \cdot \frac{dy}{dx} + 2x = 0$$

$$y\,dy + 2x\,dx = 0$$

Integrating both sides then $\int y\,dy + \int 2x\,dx = c$

$$\frac{y^2}{2} + 2\cdot\frac{x^2}{2} = c$$

$\Rightarrow y^2 + 2x^2 = c_1$ which represents a fam of concentric ellipses.

Exp. Find the orthogonal trajectories of

$$\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1,$$ where λ is a parameter.

Sol: Given Eqn is $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ is a parameter. ——— (i)

This equation is differentiating w.r.to x

$$\frac{1}{a^2} \cdot 2x + \frac{1}{a^2+\lambda} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$(a^2+\lambda) \cdot 2x + 2y \cdot a^2 \frac{dy}{dx} = 0$$

$$a^2+\lambda = -\frac{a^2 y}{x} \frac{dy}{dx} \Rightarrow \lambda = -a^2 - \frac{a^2 y}{x} \frac{dy}{dx}$$

Putting λ value in Eqn (i)

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - a^2 - \frac{a^2 y}{x} dy/dx} = 1$$

$$\frac{x^2}{a^2} - \frac{xy}{a^2} \cdot \frac{dx}{dy} = 1$$

For orthogonal trajectories, replace $y' = -\frac{1}{y'}$.

$\Rightarrow \because \frac{dy}{dx} = -\frac{dx}{dy}$ then

$$\therefore \frac{x^2}{a^2} + \frac{xy}{a^2} \frac{dy}{dx} = 1 \Rightarrow x^2 dx + xy dy = a^2 dx$$

$$x(x dx + y dy) = a^2 dx$$

$$x dx + y dy = a^2 \frac{dx}{x}$$

Integrating both sides.

$$\int x dx + \int y dy = a^2 \int \frac{dx}{x} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} = a^2 \log|x| + c$$

$$x^2 + y^2 = 2a^2 \log|x| + 2c$$

$$x^2 + y^2 = 2a^2 \log|x| + c'$$

This is required equation of the orthogonal trajectories.

Exp. Find the orthogonal trajectories of

$$r^n = a^n \cos n\theta$$

Sol. Given that $r^n = a^n \cos n\theta$ —(i)

$\Rightarrow a^n = r^n / \cos n\theta$

Diff. w.r.to θ, we get

$$n r^{n-1} \frac{dr}{d\theta} = a^n (-\sin n\theta) \cdot n$$

$$r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta \quad \text{—(ii)}$$

From (i) $a^n = r^n / \cos n\theta$ putting in Eqn (ii)

Then

$$r^{n-1} \frac{dr}{d\theta} = -r^n \frac{\sin n\theta}{\cos n\theta} \quad \text{—(iii)}$$

for orthogonal trajectories $\frac{1}{r}\frac{dr}{d\theta} = -\frac{r\,d\theta}{dr}$

$$r \frac{d\theta}{dr} = \tan n\theta$$

$$\frac{dr}{r} = \cot n\theta \cdot d\theta$$

Integrating, we get $\int \frac{dr}{r} = \int \cot n\theta\, d\theta + c$

$$\log r = \frac{1}{n} \log|\sin n\theta| + \log c$$

$\Rightarrow r^n = c^n \sin n\theta$ $\quad n \log r = \log \sin n\theta + n \log c \Rightarrow r^n = \frac{\sin n\theta}{n}$