* Bessel's formula -

This is very useful for interpolation, and it uses the differences as shown in the following table, where the brackets mean that the average of the values has to be taken.

Table - The differences

x	y	Δ	Δ^2	Δ^3	Δ^4
x_{-3}	y_{-3}				
x_{-2}	y_{-2}				
x_{-1}	y_{-1}		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
x_0	$[y_0$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-1}$
x_1	$y_1]$				
x_2	y_2				
x_3	y_3				

Hence, Bessel's formula can be assumed in the form

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots$$

$$y_p = \frac{y_0 + y_0 + \Delta y_0}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots$$

(from $\Delta y_0 = y_1 - y_0$)

$$y_p = y_0 + \left(B_1 + \frac{1}{2}\right)\Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1}$$
$$+ B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + B_5 \Delta^5 y_{-2} \cdots$$

Using the same method as in Gauss's forward formula

$$y_p = E^p y_0 = (1+\Delta)^p y_0$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\angle 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\angle 3}\Delta^3 y_0 + \cdots$$

Then we obtain

$$B_1 = p - \frac{1}{2}$$

$$B_2 = \frac{p(p-1)}{\angle 2}$$

$$B_3 = \frac{p(p-1)(p-\frac{1}{2})}{\angle 3}$$

$$B_4 = \frac{(p+1)p(p-1)(p-2)}{\angle 4}$$

$$B_5 = \frac{(p+1)p(p-1)(p-\frac{1}{2})(p-2)}{\angle 5}$$

Hence, Bessel's Interpolation formula is -

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\angle 2}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(p-\frac{1}{2})}{\angle 3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{\angle 2} \times \left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

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* Everett's formula:-

This is an extinsively used interpolation formula and use only even order differences as shown in the following table.

x_0	y_0	Δ	$\Delta^2 y_{-1}$	Δ^3	$\Delta^4 y_{-2}$	Δ^5	$\Delta^6 y_{-3}$
x_1	y_1	—	$\Delta^2 y_0$	—	$\Delta^4 y_{-1}$	—	$\Delta^6 y_{-2}$

Hence the formula has the form

$$y_p = E_0 y_0 + E_2 \Delta^2 y_{-1} + E^4 \Delta^4 y_{-2} + \cdots + F_0 y_1$$
$$+ F_2 \Delta^2 y_0 + F_4 \Delta^4 y_{-1} + \cdots$$

The coefficients $E_0, E_2, E_4 \cdots, F_0, F_2, F_4 \cdots$ can be determined by the same method as in the Stirling's & Bessel's cases and we obtained

$$E_0 = 1 - p = q$$

$$E_2 = \frac{q(q^2 - 1^2)}{\lfloor 3}$$

$$E_4 = \frac{q(q^2 - 1^2)(q^2 - 2^2)}{\lfloor 5}$$

$$F_0 = p$$

$$F_2 = \frac{p(p^2 - 1^2)}{\lfloor 3}$$

$$F_4 = \frac{p(p^2 - 1^2)(p^2 - 2^2)}{\lfloor 5}$$

Hence Everett's formula given by.

$$y_0 = q y_0 + \frac{q(q^2-1^2)}{\angle 3} \Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{\angle 5} \Delta^4 y_{-2} +$$

$$\text{------} + p y_1 + \frac{p(p^2-1^2)}{\angle 3} \Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{\angle 5} \Delta^4 y_{-1}$$

$$+ \cdots$$

where $q = 1-p$

Everett's formula truncated after second differences then its given as-

$$y_0 = (1-p) y_0 + \left[\frac{p(p-1)}{4} - \frac{(p-1)p(p-\frac{1}{2})}{6} \right] \Delta^2 y_{-1} + \cdots$$

$$+ p y_1 + \left[\frac{p(p-1)}{4} + \frac{p(p-1)(p-\frac{1}{2})}{6} \right] \Delta^2 y_0 + \cdots$$

$$y_{p} = q y_0 + \frac{q(q^2-1^2)}{\angle 3} \Delta^2 y_{-1} + \text{------} p y_1 +$$

$$\frac{p(p^2-1^2)}{\angle 3} \Delta^2 y_0$$

Exp The following table gives the values of e^{x} for certain equidistant values of x. Find the value of e^x when $x = 0.644$.

Sol. The table of differences.

	x	$y = e^x$	Δ	Δ^2	Δ^3	Δ^4
−3	0.61	1.840431				
			0.018497			
				0.000185		
−2	0.62	1.858928			0.000004	
			0.018682			−0.000004
				0.000189		
−1	0.63	1.877610			0 ($\Delta^3 y_{-1}$)	
			0.018871 (Δy_{-1})			0.000002
				0.000189 ($\Delta^2 y_{-1}$)		
0	0.64	1.896481 (y_0)			0.00002	
			0.019060 (Δy_0)			0.000001
				0.000191		
1	0.65	1.915541 (y_1)			0.000002	
			0.019251			
				0.000194		
2	0.66	1.934792	0.019445			
3	0.67	1.954237				

$x_0 = 0.64$ and $x = 0.6449$

$h = .01$ Then $p = \dfrac{0.644 - 0.64}{.01} = 0.4$

The third difference contribution is negligible in Stirling's and Bessel's Interpolation formulae. Then using Stirling's formula.

(i) $y_{(p)} = y_0 + p\,\dfrac{\Delta y_{-1} + \Delta y_0}{2} + \dfrac{p^2}{2}\Delta^2 y_{-1}$

$y(0.644) = 1.896481 + (0.4)\,\dfrac{(0.018871 + 0.019060)}{2} + \dfrac{0.16}{2}(0.000189)$

$= 1.896481 + 0.0075862 + 0.00001512$

$y(0.644) = 1.904082$

$y(0.644)$

ii) Now, using Bessel's formula

$$y_{(p)} = y_0 + p\Delta y_0 + \frac{p(p-1)}{\angle 2}\frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2}$$

$$y(0.644) = 1.896481 + (0.4)(0.019060) + \frac{(0.4)(0.4-1)}{2}\frac{(0.000189 + 0.000191)}{2}$$

$$= 1.896481 + 0.0076240 - 0.0000228$$

$$y(0.644) = 1.904082$$

iii) and Now using Everett's formula, we find

$$y_p = (1-p)y_0 + \left[\frac{p(p-1)}{4} - \frac{(p-1)p(p-\frac{1}{2})}{6}\right]\Delta^2 y_{-1} + \cdots$$

$$+ py_1 + \left[\frac{p(p-1)}{4} + \frac{p(p-1)(p-\frac{1}{2})}{6}\right]\Delta^2 y_0 + \cdots$$

$$y(0.644) = (1-0.4)(1.896481) + \left[\frac{0.4(0.4-1)}{4} - \frac{(1-0.4)(0.4)(0.4-\frac{1}{2})}{6}\right] \times (0.000189) + (0.4)(1.91554) + \left[\frac{(0.4)(0.4-1)}{4} - \frac{(1-0.4)(0.4)(0.4-\frac{1}{2})}{6}\right](0.000191)$$

$$= (0.6)(1.896481) + (-0.000012096) + 0.7662144 + (-0.000010696)$$

$$y(0.644) = 1.904082$$

All above methods, the value obtained is correct to six decimal places.