2-) Equation $f(x,y,p)=0$ is solvable for y.

Solving Eqn for y leads to

$$y = \varphi_i(x, y'), \quad i=1,2,\cdots l \quad \text{--- (1)}$$

where l is a number of solutions.

Let $p(x) = y'(x)$ is a new function of x alone.

Differentiating eqn (1) w.r.t. x then

$$p(x) = \varphi'_{ix}(x,p) + \varphi'_{ip}(x,p)\, p'(x)$$

$$p'(x) = \frac{p(x) - \varphi'_{ix}(x,p)}{\varphi'_{ip}(x,p)} \quad \text{--- (2)}$$

This differential equation is completely analogous to equation $y'(x) = f_i(x,y)$, $i=1,2 \cdots l$. --- (3)

If an explicit function $p = p(x,C)$ is a general solution of eqn (2), then by substituting this function in (1), one find a general solution of eqn (3) as an explicit function $y = \varphi_i(x, p(x,C))$.

Exp. Find the general solution of the diff. Eqn.

$$\frac{1}{x}y'^2 + xy' - y = 0$$

Solution - Given non-linear diff. Eqn.

$$\frac{1}{x}y'^2 + xy' - y = 0$$

$$y = \frac{1}{x}y'^2 + xy' \quad \text{— (i)}$$

Let $p = y' = \frac{dy}{dx}$ then $y = \frac{1}{x}p^2 + xp$ —(ii)

on diff. Eqn (ii) w.r.t. x

$$y' = \frac{1}{x}2pp' - p^2(+x^{-2}) + x.p' + p$$

$$\cancel{p} = \frac{2p}{x}p' - \frac{p^2}{x^2} + xp' + \cancel{p}$$

$$\frac{p^2}{x^2} = p'\left(\frac{2p}{x} + x\right)$$

$$\frac{p^2}{x^2} = \frac{p'}{x}(2p + x^2)$$

$$p' = \frac{p^2}{x(x^2+2p)} \quad \text{—— (iii)}$$

By substituting $p = x^2/z$, where z is a new unknown function, we obtain

$$p' = 2x.\frac{1}{z} - \frac{x^2}{z^2}.z'$$

$$\frac{2x}{z} - \frac{x^2}{z^2}.z' = \frac{x^{\cancel{3}4}}{x.z^2(x^2 + 2.\frac{x^2}{z})}$$

$$\frac{2x}{z} - \frac{x^2}{z^2}.z' = \frac{x^{\cancel{3}4}}{z(x^2z + 2x^2)} = \frac{x^3}{z(z+2)x^2} = \frac{x}{z(z+2)}$$

$$\frac{2x}{z} - \frac{x^2}{z^2}z' = \frac{x}{z(z+2)}$$

$$\frac{x^2}{z^2}z' = \frac{-x}{z(z+2)} + \frac{2x}{z}$$

$$\frac{x^2}{z^2}z' = \frac{2x(z+2) - x}{z(z+2)}$$

$$\frac{x^2}{z^2}z' = \frac{2xz + 4x - x}{z(z+2)}$$

$$\frac{x^2}{z^2}z' = \frac{2xz + 3x}{z(z+2)}$$

$$\frac{x}{z}z' = \frac{2z+3}{(z+2)}$$

$$xz' = \frac{z(2z+3)}{z+2}$$

using partial fraction and integration

$$\frac{dx}{x} = \frac{dz\,(z+2)}{z(2z+3)} = \frac{(z+2)\,dz}{z(2z+3)}$$

$$\int \frac{2}{3z}dz - \int \frac{1}{3(2z+3)}dz = \log|x| + \log|c|$$

$$\frac{2}{3}\log|z| - \frac{1}{3}\cdot\log(2z+3)\cdot\frac{1}{2} = \log|xc|$$

$$\frac{2}{3}\log z - \frac{1}{6}\log(|2z+3|) = \log|xc|$$

Substituting for p then

$$x^2 = \frac{3c^2p^4}{1-2c^6p^3} \quad \text{is } \underline{g.s.}$$