(2) Inhomogeneous (non-homogeneous) Equation:

* The Complete Equations with constant Coefficients. Method of Underdetermined Coefficients. The non-homogeneous-nth-order linear equation:

$$a_n y^n + a_{n-1} y^{n-1} + \cdots a_1 y' + a_0 = f(x) \quad \text{—— (1)}$$

A particular solution of the non-homogeneous n-th order linear differential equation with constant coefficients can be obtained by the method of undetermined coefficients provided that $f(x)$ is of an appropriate form.

(i) First consider the case that $f(x)$ is polynomial of degree m.

$$f(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0$$

where $b_0, b_1, \cdots b_m$ are given constants. —— (ii)

It is natural to look for a particular solution of the form.

$$Y_p = A_m x^m + A_{m-1} x^{m-1} + \text{——} A_1 x + A_0$$

Substituting $y, y', y'' \cdots$ in Eqn of n-th order homogeneous linear differential equation (i) and equating the coefficients of like powers of x from its and eqn (ii), we find form the terms in x^m that

$$a_n A_m = b_m.$$

Provided that $a_n \neq 0$ we have $A_0 = b_0/a_0$

The constants $A_1, A_2 \dots A_m$ are determined from the coefficients of the terms $x, x^2, \dots x^{m-1}, x^m$

More generally if zero is an s-fold root of the auxiliary polynomial $(a_n = a_{m-1} = \dots a_{n-s+1} = 0)$ then a suitable form for $y_p(x)$ is

$$y_p(x) = x^s(A_m x^m + A_{m-1}x^{m-1} + \dots A_0)$$

Exp. Solve $y'' - y = x^2 + 1$

Solu. First we solve general solution of homogeneous linear equation.

$$y'' - y = 0 \quad \text{--- (i)}$$

The auxiliary equation

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$m_1 = 1, \ m_2 = -1$$

$$y_0 = c_1 e^x + c_2 e^{-x} \quad \text{--- (ii)}$$

Since $f(x)$ is a polynomial of the second degree we look for a particular solution of the form

$$y_p = A_2 x^2 + A_1 x + A_0 \quad \text{--- (iii)}$$

$$y_p' = 2A_2 x + A_1$$

$$y_p'' = 2A_2$$

On substituting y_p'' and y_p into the original equation $y'' - y = x^2+1$

then

$$2A_2 - A_2x^2 - A_1x - A_0 = x^2+1$$

Equating coefficients of like terms, we have

$$-A_2 = 1 \Rightarrow A_2 = -1.$$

$$A_1 = 0$$

$$2A_2 - A_0 = 1 \Rightarrow A_0 = -3$$

$A_2 = -1$, $A_1 = 0$ & $A_0 = -3$ putting in Eqn (ii)

$$Y_p = -x^2 + 0 \cdot x - 3$$

$$Y_p = -x^2 - 3$$

Hence, complete solution is

$$Y = y_c + Y_p$$

$$y = c_1e^{x} + c_2e^{-x} - x^2 - 3$$

(ii) As a second case suppose that $f(x)$ is of the form

$$f(x) = e^{\alpha x}(b_m x^m + b_{m-1}x^{m-1} + \cdots b_0)$$

Then we try y_p in the form

$$y_p = e^{\alpha x}(A_m x^m + A_{m-1}x^{m-1} + \cdots A_0)$$

provided that $e^{\alpha x}$ is not a solution of the homogeneous equation.

* If α is an s-fold root of the auxiliary equation a suitable form for y_p is

$$y_p = x^s e^{\alpha x}(A_m x^m + A_{m-1}x^{m-1} + \cdots A_0)$$

Exp. Solve $y'' - y = e^{3x}$

Solution Give that second-order non-homogeneous linear equation

$$y'' - y = e^{3x} \quad \text{——— (i)}$$

Homogeneous Eqn is $y'' - y = 0$ ——(ii)

and $f(x) = e^{3x}$

The auxiliary Eqn. $m^2 - 1 = 0$ then

$$m = \pm 1$$

Hence $y_0 = C_1 e^{x} + C_2 e^{-x}$ ——— (iii)

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Here e^{x} and e^{-x} are solutions of the homogeneous equation. Since e^{3x} is not a solution of the homogeneous equation, we try a particular solution of the form

$$y_p = A_0 e^{3x} \quad \text{——} \ (4)$$

Here $P_n(x)$ is zero degree of x.

$$y_p' = A_0 \cdot e^{3x} \cdot 3 = 3A_0 e^{3x}$$

$$y_p'' = 3A_0 e^{3x} \cdot 3 = 9A_0 e^{3x}$$

on substituting y_p'' and y_p in Eqn (i) and equating terms of coefficients.

$$9A_0 e^{3x} - A_0 e^{3x} = e^{3x}$$

$$(9A_0 - A_0) e^{3x} = e^{3x}$$

$$8A_0 = 1 \quad A_0 = 1/8$$

Putting A_0 in Eq (iv)

$$y_p = \frac{1}{8} e^{3x}$$

The Complete solution is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{8} e^{3x}$$

Another way of previous Example:

Given Eqn $y'' - y = e^{3x}$

or $\frac{d^2y}{dx^2} - y = e^{3x}$ $\quad \because \frac{d}{dx} \to D$

$(D^2 - 1)y = e^{3x}$ $\quad \frac{d^2}{dx^2} \to D^2$

$\vdots$

The auxiliary Equation $D^2 - 1 = 0$ $\quad \frac{d^n}{dx^n} \to D^n$

$\Rightarrow D = \pm 1$

$C.F = y_0 \Rightarrow c_1 e^{x} + c_2 e^{-x}$ is complementary function.

Particular solution by particular integral

$$P.I = \frac{1}{D^2 - 1} e^{3x} = e^{3x} \frac{1}{(3)^2 - 1} = \frac{1}{8} e^{3x}$$

Hence the complete solution is $C.F + P.I$

$$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{8} e^{3x}$$

Ans