Matrix of Linear Transformation

Matrix Representation of a Linear Transformation

Let U be an m-dimensional vector space over the field F and let V be an n-dimensional vector space over the field F.

Let $S_1 = \{x_1, x_2, x_3 \cdots x_m\}$ be basis of U and
$S_2 = \{y_1, y_2, y_3, \cdots y_n\}$ be basis of V i.e. $\dim U = m$
and $\dim V = n$ respectively.

If T is linear transformation from U into V, then $T(x_1), T(x_2) \cdots T(x_m)$ are vectors of V, and $S_2 = \{y_1, y_2, \cdots y_n\}$ is basis of V so that each $T(x_i)$ is linear combination of the elements of S_2.

$$T(x_1) = \alpha_{11}y_1 + \alpha_{12}y_2 + \cdots \alpha_{1n}y_n \qquad \alpha_{ij} \in F$$

$$1 \le i \le m$$

$$1 \le j \le n$$

$$T(x_2) = \alpha_{21}y_1 + \alpha_{22}y_2 + \cdots \alpha_{2n}y_n$$

$$\vdots$$

$$T(x_m) = \alpha_{m1}y_1 + \alpha_{m2}y_2 + \cdots \alpha_{mn}y_n$$

Then the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The transpose of A is called the matrix of T relative to ordered bases S_1 and S_2.

The Transpose of the above matrix denoted by $[T]_{S_2, S_1}$ is called the matrix representation of T relative to the ordered basis S_1

$$[T]_{S_1} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}_{n \times m}$$

where $O(S_1) = m$
$O(S_2) = n$

Exp. Let $T: U \to V$ be a linear transformation where $\dim U = 2$ and $\dim V = 3$. let S_1 and S_2 be ordered basis of U and V, respectively.

(I) The order of $[T]_{S_1,S_2}$ is 3×2 for $\dim U = 2$, $\dim V = 3$

(II) The order of $[T]_{S_2,S_1}$ is 2×3 for $\dim V = 3$, $\dim U = 2$

Exp. Let T be a linear transformation from
$T: U(R^3) \to V(R^2)$ defined as
$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$
if $B = S_1 = \{(1,0,-1), (1,1,1), (1,0,0)\}$ $\dim U = 3$
$B' = S_2 = \{(0,1), (1,0)\}$ be ordered bases of $\dim V = 2$
R^3 and R^2, respectively, then find the matrix of T relative to S_1, S_2. Also find rank (T) and nullity (T).

__Solution__ Using condition for $T: U(R^3) \to V(R^2)$ fr

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$$

$$T(1,0,-1) = (1+0, -2\times1 - 1) = (1,-3)$$
$$T(1,1,1) = (1+1, 2\times1 - 1) = (2,1) \Rightarrow A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$
$$T(1,0,0) = (1+0, 2\times0 - 1) = (1,-1)$$

further-
$$T(0,1,-1) = (1,-3) = a(0,1) + b(1,0) \Rightarrow (0,a) + (b,0) = (0+b, a+0)$$
$$\Rightarrow (b,a)$$
$$b = 1 \text{ \& } a = -3$$

$$T(0,1,-1) = (1,-3) = -3(0,1) + 1(1,0)$$
$$T(1,1,1) = (2,1) = 1(0,1) + 2(1,0) \Rightarrow A \begin{bmatrix} -3 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$$
$$T(1,0,0) = (1,-1) = -1(0,1) + 1(1,0)$$

$$[T]_{S_1,S_2} = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$