Def. of Linear Transformation – Let V and W be vector spaces.

The function $T: V \to W$ is called linear transformation of V into W if the following two conditions/properties are true/hold as-

(i) $T(u+v) = T(u) + T(v), \quad \forall\, u, v \in V$

(ii) $T(\alpha v) = \alpha T(v) \qquad v \in V$ and $\alpha \in F$ is a scalar.

In other word, the conditions (i) & (ii) can be combined into a single condition:

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

where $\alpha, \beta \in F$ & $u, v \in V$

Linear transformation (T) is also known as homomorphism of V into W.

If T is a homomorphsim of V onto W then W is called the homomorphic image of V.

Exp. For any vector $V = (v_1, v_2)$ in R^2

let $T: R^2 \to R^2$ be defined by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) find the image of $V = (-1, 2)$

(b) find the preimage of $w = (-1, 11)$

Sol. (a) for $V = (-1, 2)$

we have $T: R^2 \to R^2$ as

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

$$T(-1, 2) = (-1 - 2, -1 + 2 \times 2)$$

$$T(-1, 2) = (-3, 3)$$ is the image of $(-1, 2)$

(b) If $T(v) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$

Then $v_1 - v_2 = -1$ —— (i)

$v_1 + 2v_2 = 11$ —— (ii)

from (ii) − (i)

$3v_2 = 12$

$v_2 = 4$

and $v_1 = 3$

$v = (3, 4)$

$w = (-1, 11)$

So the preimage of $w = (-1, 11)$ is $v = (3, 4)$ in $\mathbb{R}^2$

* Hence, functions (from one vector space to another) that preserve the operations of vectors addition and scalar multiplication, are called linear transformations.

Exp The mapping $T : V(R^3) \to V(R^2)$ defined as $T(x_1, x_2, x_3) = (x_1, x_2)$ is homomorphism (linear Transformation) of $V(R^3)$ onto $V_2(R^2)$ or $V_3(R)$ onto $V_2(R)$

Solution Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ belong to $V_3(R)$ i.e $x, y \in V_3(R)$

Also α & $\beta \in (R)$ and condition $T(x_1, x_2, x_3) = (x_1, x_2)$

We have Linear Transformation ——①

How

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$\therefore T(\alpha x + \beta y) = T[\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)]$$

$$= T[(\alpha x_1, \alpha x_2, \alpha x_3) + (\beta y_1, \beta y_2, \beta y_3)]$$

$$= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \text{ by cond ①}$$

$$\therefore T(\alpha x + \beta y) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \text{ ——②}$$

How $\alpha T(x) + \beta T(y) = \alpha T(x_1, x_2, x_3) + \beta T(y_1, y_2, y_3)$

$$= \alpha(x_1, x_2) + \beta(y_1, y_2)$$

—— by cond ①

$$= (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$$

$\alpha T(x) + \beta T(y) = (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$

$\alpha T(x) + \beta T(y) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$ —— (3)

from (2) & (3)

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$\therefore$ Hence $T: V(R^3) \to V(R^2)$ is Linear Transformation or homomorphism for condition $T(x_1, x_2, x_3) = (x_1, x_3)$

Exp. Some functions that are not linear transformations.

(i) $f(x) = \sin x$ or $\cos x$ or $\tan x$

(ii) $f(x) = x^2$

(iii) $f(x) = x + 1$ or $x + c$ where c is constant

Sol. (a) $f(x) = \sin x$ is not a L.T. from $V(R)$ into $V(R)$ because, in general

$\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$

for instance $\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin \pi/2 + \sin \pi/3$

(b) $f(x) = x^2$ is not a linear transformation from R into R because, in general

$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$

i.e. $(1+3)^2 \neq 1^2 + 3^2$

(c) $f(x)=x+1$ is not a linear transformation from R into R because, in general

$$f(x_1+x_2)=x_1+x_2+1$$

whereas

$$f(x_1)+f(x_2)=x_1+1+x_2+1$$

$$=x_1+x_2+2$$

$$\therefore f(x_1+x_2)\neq f(x_1)+f(x_2)$$

* In this function points out two uses of the term linear. In calculus $f(x)=x+1$ is called a linear function because its graph is a line. It is not a linear transformation from the vector space R into R, however, becaus it preserves neither vector addition not scalor multiplication.