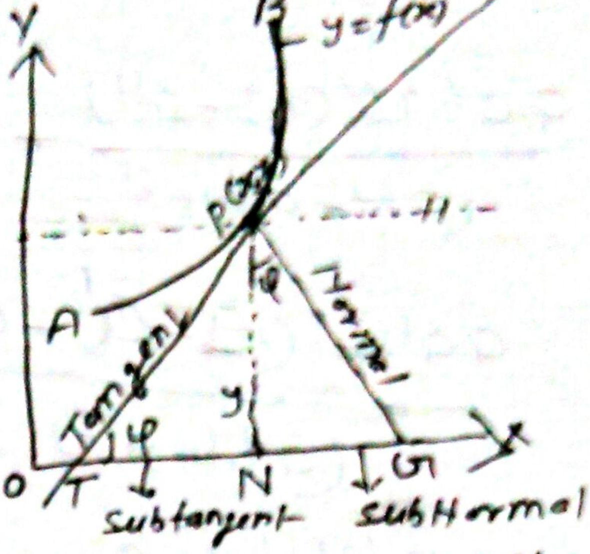
Length of Normal, Subnormal, Tangents and Subtangents:

Let the equation of the curve AB be $y = f(x)$. Let P be any point on the curve whose coordinate P(x, y).



PN is perpendicular on x axis (OX) from P. Then
ON = x & PN = y and
PT is tangent and PG is normal at point P of the
curve which meets the x axis at T and G respectively.

Then the length TN (i.e the projection of the tangent PT on the x-axis) is called subtangent and NG (i.e the projection of the normal PG on the ~~tangents~~) is called subnormal PG.

So, we want to find the lengths of the subtangent TN, subnormal NG, tangent PT and normal PG.

Let the tangent PT make an angle φ with the x-axis. $\angle PTN = \varphi$

$\therefore \angle NPG = \varphi$

(i) In $\triangle TNP$ $\tan\varphi = \frac{y}{TN}$ $\Rightarrow$ $TN = y/\tan\varphi$

$\because \tan\varphi = \frac{dy}{dx} \Rightarrow$ slope

$TN = y / dy/dx$

$TN = y\frac{dx}{dy}$

This is the length of subtantent (TN) $= y/\frac{dy}{dx}$ or $y\frac{dx}{dy}$

(ii) Again, from ΔPNG

$$\tan\psi = \frac{NG}{PN} = \frac{NG}{y} \Rightarrow NG = y\tan\psi = y.\frac{dy}{dx}$$

Hence, the length of the subnormal $(NG) = y\frac{dy}{dx}$

(iii) From ΔPNT

$$PT^2 = PN^2 + TN^2$$

$$PT^2 = y^2 + \left(y/\frac{dy}{dx}\right)^2 = y^2 + y^2.\left(\frac{dx}{dy}\right)^2$$

$$PT^2 = y^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]$$

$$PT = y\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}$$

$$\text{or } PT = \frac{y}{\frac{dy}{dx}}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$\therefore$ The length of the ~~sub~~ tangent $PT = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$

(iv) Also, from ΔPNG

$$PG^2 = PN^2 + NG^2$$

$$= y^2 + y^2\left(\frac{dy}{dx}\right)^2$$

$$PN^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$PN = y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$\therefore$ The length of the normal $PG = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

—x—

Q. Show that in the curve $y = b e^{x/a}$, then The sub-tangent is constant and subnormal is $\frac{y^2}{a}$

Sol. Given curve

$$y = b e^{x/a}$$

Diff. w.r.t. x, we get

$$\frac{dy}{dx} = b \cdot e^{x/a} \cdot \frac{1}{a} = \frac{b}{a} e^{x/a}$$

The length of subtangent $= \frac{y}{\frac{dy}{dx}}$

$$\Rightarrow \quad = \frac{b e^{x/a}}{\frac{b}{a} e^{x/a}} = \frac{1}{1/a} = a = \text{const.}$$

The length of subnormal $= y \cdot \frac{dy}{dx}$

$$= b e^{x/a} \cdot \frac{b}{a} e^{x/a}$$

$$= \frac{1}{a} \cdot b^2 e^{2x/a}$$

$$= \frac{1}{a} (b e^{x/a})^2 \qquad \because y = b e^{x/a}$$

$$= \frac{y^2}{a}$$