

* Lagrange's Equation:-

An equation of the form-

$$y = x f(y') + g(y')$$

or $y = x f(p) + g(p)$ — (i)

where $p = y' = \frac{dy}{dx}$ is called Lagrange's equation

Equation (i) is reduced to a linear equation in x by differentiating and taking into consideration that $dy = p dx$.

Then

$$p dx = f(p) dx + [x \cdot f'(p) + g'(p)] dp \text{ — (ii)}$$

if $p \neq f(p)$, then from (i) and (ii) we get the general solution in parametric form:-

From Eqn (ii)

$$\frac{dx}{dp} (p - f(p)) - x f'(p) = g'(p)$$

$$\frac{dx}{dp} - x \frac{f'(p)}{p - f(p)} = \frac{g'(p)}{p - f(p)} \text{ — (iii)}$$

where $p - f(p) \neq 0$. Hence a general solution of the Lagrange's equation can be written

$$x = x(p, c), \quad y = x f(p) + g(p)$$

where p is parameter and c is an arbitrary constant.

$$\text{if } x = c f_1(p) + g_1(p)$$

$$\Rightarrow y = [c f_1(p) + g_1(p)] f(p) + g(p)$$

In those points p_i^* , where $p_i^* - f(p_i^*) = 0$, we have the singular solutions as the straight lines

$$y = x f(p_i^*) + g(p_i^*)$$

where $i = 1, 2, \dots, k$, k is number of the singular solutions.

Exp. Find the general solution of the Lagrange eqn

$$x y'^2 - y' - y = 0$$

Solution Given that

$$x y'^2 - y' - y = 0$$

$$y = x p^2 - p \quad \therefore y' = \frac{dy}{dx} = p$$

$$dy = p dx$$

Eph (1) differentiate with respect to x

$$\frac{dy}{dx} = p^2 + x \cdot 2p \cdot \frac{dp}{dx} - \frac{dp}{dx}$$

$$p = p^2 + (2xp - 1) \frac{dp}{dx}$$

$$\frac{dp}{dx} (1 - 2xp) = p(p - 1)$$

$$1 - 2xp = p(p - 1) \frac{dx}{dp}$$

$$\frac{dx}{dp} = \frac{1}{p(p-1)} - \frac{2x}{p-1}$$

$$\frac{dx}{dp} + \frac{2x}{p-1} = \frac{1}{p(p-1)}$$

This is in Linear differential Equation

x form if $p \neq 0$ and $p \neq 1$

$$I.F = e^{\int \frac{2}{p-1} dp}$$

$$x \cdot e^{\int \frac{2}{p-1} dp} = \int \frac{1}{p(p-1)} \cdot e^{\int \frac{2}{p-1} dp} + c$$

$$x \cdot e^{2 \log(p-1)} = \int \frac{1}{p(p-1)} \cdot (p-1)^2 dp + c$$

$$x \cdot (p-1)^2 = \int \left(1 - \frac{1}{p}\right) dp + c$$

$$x(p-1)^2 = p - \log|p| + c$$

$$x = \frac{c + p - \log|p|}{(p+1)^2} \quad \text{where } c \text{ is arbitrary constant.}$$

$$y = xp^2 - p$$

is the general solution of the given equation
if $p=0$ or $p=1$ we find two singular solutions
 $y=0$ and $y=x-1$, respectively.

Exp. Solve the equation

$$y = 2y'x + \frac{1}{y'}$$

Sol. Given that $y = 2y'x + \frac{1}{y'}$ — (1)

$$\text{Putting } y' = p \Rightarrow dy = p dx$$

$$\text{Then } y = 2px + \frac{1}{p}$$

$$\frac{dy}{dx} = 2p + 2x \frac{dp}{dx} - \frac{1}{p^2} \frac{dp}{dx}$$

$$dy = 2p dx + 2x dp - \frac{1}{p^2} dp$$

$$p dx = 2p dx + 2x dp - \frac{dp}{p^2}$$

$$p dx = 2p dx + 2x dp - \frac{dp}{p^2}$$

$$\frac{dx}{dp} = -\frac{2}{p}x + \frac{1}{p^3} \Rightarrow \frac{dx}{dp} + \frac{2}{p}x = \frac{1}{p^3}$$

This is linear differential Eqn in x

$$I.F. = e^{\int +2/p dp} = e^{+2 \log p} = \frac{p^2}{p}$$

$$x \cdot \left(\frac{p^2}{p^2} \right) = \int \frac{1}{p^3} \cdot \left(\frac{p^2}{p^2} \right) dp + C$$

$$= \int \frac{1}{p} dp + C$$

$$x(p^2) = \log |p| + C$$

$$x = \frac{1}{p^2} [\log(p) + C]$$

$$y = 2px + \frac{1}{p}$$

To find the singular solution

$$\frac{2x}{p} - \frac{1}{p^3} = 0 \Rightarrow x = \frac{1}{2p^2}$$

$$y = \frac{2}{p}$$

$$y = \pm 2\sqrt{2x}$$

* Clairaut's Equation:-

An equation of the form

$$y = x y' + f(y')$$

$$\text{or } y = px + f(p) \quad (\text{where } y' = p) \text{ --- (i)}$$

is known as Clairaut's equation.

(*) Notice that this is a particular case of Lagrange equation if $f(y') = y' \Rightarrow (p = \frac{dy}{dx})$

Now differentiating w.r.to x in Eq (i)

$$y = px + f(p) \text{ --- (i)}$$

$$\frac{dy}{dx} = p \cdot 1 + x \cdot \frac{dp}{dx} + \frac{d}{dx}(f(p))$$

$$p = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}$$

$$[x + f'(p)] \frac{dp}{dx} = 0$$

$$\Rightarrow \frac{dp}{dx} = 0 \quad \therefore p = c \text{ --- (ii)}$$

Putting in Eq (i) $p = c$ then

$$y = cx + f(c)$$

as the general solution of Eq (i).

Hence, Clairaut's equation solution is obtained on replacing p by c .

Exp. Find the general solution of the Clairaut's equation

$$y = xy' + e^{y'}$$

Sol. Given equation

$$y = xy' + e^{y'} \quad \text{--- (i)}$$

put $y' = p$ then we get

$$y = px + e^p \quad \text{--- (ii)}$$

This is in Clairaut's Eqn form

Since $f(p) = e^p$

We have the general solution of this Clairaut's equation (ii) replacing p by c

then

$$y = cx + e^c$$

where c is arbitrary constant

The singular solution is

$$x = -\frac{1}{e^p}, \quad y = xp + e^p$$

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