Isomorphic Vector Space: A vector space $U(F)$ is said to be isomorphic too vector space $V(F)$, If there exists a mapping $T: U \to V$ such that

(i) T is linear transformation i.e $T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)$

(ii) T is one-to-one i.e $T(x) = T(y) \Rightarrow x = y$ $\quad x, y \in U$

(iii) T is onto i.e for each $v \in V$, there exists some $u \in U$ $T(u) = v$, T is onto iff $V = T(U)$

Also then the two vector spaces U and V are said to be isomorphic and symbollically write as $U(F) \cong V(F)$.

Theorem - Any two finite-dimensional vector spaces over the same field are isomorphic.

Proof - Let $U(F)$ and $V(F)$ be two finite dimensional vectors spaces over field F such that $\dim U = \dim V = n$

Then prove that $U(F) \cong V(F)$

Let the sets of vectors $\{x_1, x_2, \ldots x_n\}$ $\{u_1, u_2, \ldots u_n\}$ and $\{y_1, y_2, y_3, \ldots y_n\}$ $\{v_1, v_2, v_3, \ldots v_n\}$ be the bases of U and V respectively.

Any vector $u \in U$ canbe expressed as

$u = \alpha u_1 + \alpha u_2 + \ldots \alpha_n u_n$

$u = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$ $\quad \alpha_i \in F$ & $u \in U$

$u' = \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_n x_n$ $\quad \beta_i \in F$ & $u' \in U$

Let $T: U \to V$ be defined by as

$$T(u) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots \alpha_n y_n$$

Since in the expression for u as a linear combination of $x_1, x_2, x_3, \ldots x_n$ with the scalars $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are unique therefor, the mapping T is well-defined i.e $T(u)$ is a unique element of V.

(i) T is one-one. We have $T(u) = T(u') \Rightarrow u = u'$

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n) = T(\beta_1 x_1 + \beta_2 x_2 + \ldots \beta_n x_n)$$

$\alpha_i, \beta_i \in F$

$\alpha_1 y_1 + \alpha_2 y_2 + \ldots \alpha_n y_n = \beta_1 y_1 + \beta_2 y_2 + \ldots \beta_n y_n$ $\quad x_i \in U$

$(\alpha_1 - \beta_1) y_1 + (\alpha_2 - \beta_2) y_2 + \ldots (\alpha_n - \beta_n) y_n = 0$

$\alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0$ $\ldots$ $\alpha_n - \beta_n = 0$, $\because y_1, y_2 \ldots y_n$ are L.I.

$\Rightarrow \alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ $\ldots$ $\alpha_n = \beta_n$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n = \beta_1 x_1 + \beta_2 x_2 + \cdots \beta_n x_n$

$\therefore f$ is one-one

(ii) <u>T is onto V.</u>

if $v = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \cdots \alpha_n y_n$ is any element of V, then there exists ($\exists$) an element $u = \alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n \in U$ such that $T(u) = v$

$T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n) = \alpha_1 y_1 + \alpha_2 y_2 + \cdots \alpha_n y_n$

$\therefore$ T is onto V.

(iii) <u>T is a Linear transformation</u> - We have

$T\{\alpha(\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n) + \beta(\beta_1 x_1 + \beta_2 x_2 + \cdots \beta_n x_n)\}$

$T\{(\alpha\alpha_1 x_1 + \alpha\alpha_2 x_2 + \cdots \alpha\alpha_n x_n) + (\beta\beta_1 x_1 + \beta\beta_2 x_2 + \cdots \beta\beta_n x_n)\}$

$T\{(\alpha\alpha_1 + \beta\beta_1)x_1 + (\alpha\alpha_2 + \beta\beta_2)x_2 + \cdots (\alpha\alpha_n + \beta\beta_n)x_n\}$

$(\alpha\alpha_1 + \beta\beta_1)y_1 + (\alpha\alpha_2 + \beta\beta_2)y_2 + \cdots (\alpha\alpha_n + \beta\beta_n)y_n$

$\alpha(\alpha_1 y_1 + \alpha_2 y_2 + \cdots \alpha_n y_n) + \beta(\beta_1 y_1 + \beta_2 y_2 + \cdots \beta_n y_n)$

$\alpha T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n) + \beta T(\beta_1 x_1 + \beta_2 x_2 + \cdots \beta_n x_n)$

$\alpha T(u) + \beta T(u')$

$\therefore$ T is a linear transformation.

Hence T is an isomorphism of U onto V

$\therefore U \cong V$

<u>Definition</u> - A Linear transformation $T: U \to V$ is called an isomorphism if T is one-one and onto. i.e $T(u) = T(u') \Rightarrow u = u'$; $u, u' \in U$.
and $T(u) = v$ $v \in V$