Q.6 In the curve $x^{m+n} = a^{m+n} y^{2n}$, prove that the mth power of the subtanget varies as the nth power of the subnormal.

Sol. Given the curve $x^{m+n} = a^{m+n} y^{2n}$ —— (1)

$$y^{2n} = \frac{1}{a^{m+n}} \cdot x^{m+n}$$

Diff. w.r.t. x, we get

$$2n \cdot y^{2n-1} \frac{dy}{dx} = \frac{1}{a^{m+n}} \cdot (m+n)\, x^{m+n-1}$$

$$\frac{dy}{dx} = \frac{m+n}{2n} \cdot \frac{1}{a^{m+n}} \cdot \frac{x^{m+n} \cdot x^{-1}}{y^{2n} \cdot y^{-1}}$$

$$\frac{dy}{dx} = \frac{m+n}{2n} \cdot \frac{1}{\cancel{a^{m+n}}} \cdot \cancel{a^{m+n}} \cdot \frac{x^{-1}}{y^{-1}}$$

$$\frac{dy}{dx} = \frac{m+n}{2n} \cdot \frac{y}{x} \quad \text{—— (ii)}$$

Now, the length of subtangent (ST) $= y \Big/ \frac{dy}{dx}$

$$ST = \cancel{y} \Big/ \frac{m+n}{2n} \cdot \frac{\cancel{y}}{x}$$

$$ST = \frac{2nx}{m+n}$$

mth power of subtangent $(ST)^m$

$$(ST)^m = \left(\frac{2nx}{m+n}\right)^m \quad \text{—— (3)}$$

The length of subnormal (SN) = $y \cdot \frac{dy}{dx}$

$$SN = y \cdot \frac{m+n}{2n} \cdot \frac{y}{x}$$

$$SN = \frac{m+n}{2n} \frac{y^2}{x}$$

n^{th} power of subnormal is $(SN)^n$

$$(SN)^n = \left(\frac{m+n}{2n} \frac{y^2}{x}\right)^n$$

$$\frac{(ST)^m}{(SN)^n} = \frac{(2nx/m+n)^m}{[(m+n)y^2/2nx]^n}$$

$$= \frac{(2n)^m \cdot x^m}{(m+n)^m} \times \frac{(2n)^n \cdot x^n}{(m+n)^n \cdot y^{2n}}$$

$$= \frac{(2n)^{m+n} \cdot x^{m+n}}{(m+n)^{m+n} \cdot y^{2n}}$$

$$= \frac{(2n)^{m+n}}{(m+n)^{m+n}} \cdot a^{m+n} \quad \text{from (1)}$$

$$= \left(\frac{2na}{m+n}\right)^{m+n}$$

$$\frac{(ST)^m}{(SN)^n} = \text{const.}$$

$$(ST)^m \propto (SN)^n$$

Hence, m^{th} power of subtangent varies as n^{th} power of subnormal.

Exp. Show that in the curve $by^2 = (a+x)^3$, the square of the subtangent varies as the subnormal.

Sol. Given curve

$$by^2 = (a+x)^3$$

Differentiating the curve w.r.t. x, we get

$$b \cdot 2y \cdot \frac{dy}{dx} = 3(a+x)^2 \cdot 1$$

$$\frac{dy}{dx} = \frac{3}{2by} \cdot (a+x)^2$$

Now, the length of subtangent $= \dfrac{y}{dy/dx}$

$$ST = \frac{y}{3(a+x)^2/2by}$$

$$= \frac{2by^2}{3(a+x)^2} = \frac{2b}{3} \cdot \frac{y^2}{(a+x)^2} \quad \text{—— (I)}$$

And, the length of subnormal $= y \cdot \dfrac{dy}{dx}$

$$SN = \cancel{y} \cdot \frac{3}{2b\cancel{y}} (a+x)^2$$

$$= \frac{3}{2b} (a+x)^2 \quad \text{—— (II)}$$

$\Rightarrow$ Square of the subtangent varies as the subnormal

$$\therefore \frac{(y/dy/dx)^2}{y \cdot \frac{dy}{dx}} = \frac{\left[\frac{2b}{3} \cdot \frac{y^2}{(a+x)^2}\right]^2}{\frac{3}{2b} (a+x)^2}$$

$$\frac{(ST)^2}{SN} = \frac{4b^2}{9} \frac{y^4}{(a+x)^4} \times \frac{2b}{3} \cdot \frac{1}{(a+x)^2}$$

$$\frac{(ST)^2}{SH} = \frac{8b^3}{27} \cdot \frac{y^4}{(a+x)^6} = \frac{8\cancel{b^3}}{27} \cdot \frac{\cancel{y^4}}{\cancel{b^2}\cancel{y^4}}$$

$\because by^2 = (a+x)^3$

$$\frac{(ST)^2}{SH} = \frac{8b}{27} = \text{constant}$$

$\therefore ST^2 \propto SH.$

$(\text{Subtangent})^2 \propto \text{subnormal}$

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