Q.3

Exp. For the curve $x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

Prove that (i) $\varphi = \theta/2$, (ii) $\frac{ds}{dy} = \sqrt{\frac{2a}{y}}$

(iii) $\frac{ds}{d\theta} = 2a\cos\theta/2$

Sol. Given curve that

$x = a(\theta + \sin\theta)$ & $y = a(1 - \cos\theta)$

Differentiating this w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(0 - (-\sin\theta)) = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = a\sin\theta \cdot \frac{1}{a(1+\cos\theta)}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2 \cdot \cos\theta/2}{\cancel{1} + 2\cos^2\theta/2 - \cancel{1}}$$

$$\frac{dy}{dx} = \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

$$\Rightarrow \tan\varphi = \tan\theta/2 \Rightarrow \varphi = \theta/2$$ (i) is prove

Now $$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{\cos\theta/2}{\sin\theta/2}\right)^2}$$

$$= \sqrt{\frac{\sin^2\theta/2 + \cos^2\theta/2}{\sin^2\theta/2}}$$

$\because \frac{dy}{dx} = \tan\theta/2$

$\frac{dx}{dy} = \cot\theta/2$

$= \frac{\cos\theta/2}{\sin\theta/2}$

$$\frac{ds}{} = \sqrt{1/\sin^2\theta/2} = \operatorname{cosec}\theta/2$$

We have $y = a(1-\cos\theta)$

$$y = a(\cancel{1} - \cancel{1} + 2\sin^2\theta/2)$$

$$y = 2a.\sin^2\theta/2$$

$$\operatorname{cosec}\theta/2 = \sqrt{\frac{2a}{y}}$$

$$\Rightarrow \quad \frac{ds}{dy} = \sqrt{\frac{2a}{y}} \quad \text{proved (ii)}$$

for (iii) $\frac{ds}{d\theta} = 2a\cos\theta/2$

We know

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1+\cos^2\theta + 2\cos\theta + \sin^2\theta]$$

$$= a^2[\underline{\cancel{\sin^2\theta} + \cos^2\theta} + 2\cos\theta + 1]$$

$$= 2a^2[1+\cos\theta]$$

$$= 2a^2[\cancel{1} + 2\cos^2\theta/2 - \cancel{1}]$$

$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2\cos^2\theta/2$$

$$\frac{ds}{d\theta} = 2a\cos\theta/2$$

Hence proved.

Q.2 Prove that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

if $x = a\sin\phi$, $\frac{ds}{d\phi} = a\sqrt{1-e^2\sin^2\phi}$

$\because e = c/a$

$\left(\because e^2 = 1-\frac{b^2}{a^2}\right)$

$(ae)^2 = a^2-b^2$

Sol. Given that

$$x = a\sin\phi \quad \text{— ①}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{\cancel{a^2}\sin^2\phi}{\cancel{a^2}}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2} = 1-\sin^2\phi = \cos^2\phi$$

$$y^2 = b^2\cos^2\phi \Rightarrow y = b.\cos\phi \quad \text{— ⑪}$$

We have

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2$$

from ① & ⑪ Diff. w.r.t. ϕ.

$$\frac{dx}{d\phi} = a\cos\phi$$

$$\frac{dy}{d\phi} = -b\sin\phi$$

$$\left(\frac{ds}{d\phi}\right)^2 = (a\cos\phi)^2 + (-b\sin\phi)^2$$

$$= a^2\cos^2\phi + b^2\sin^2\phi$$

$$= a^2 - a^2\sin^2\phi + b^2\sin^2\phi$$

$$\left(\frac{ds}{d\phi}\right)^2 = a^2-(a^2-b^2)\sin^2\phi$$

$$\left(\frac{ds}{d\phi}\right)^2 = a^2\left(1-\left(1-\frac{b^2}{a^2}\right)\sin^2\phi\right)$$

$$\left(\frac{ds}{d\phi}\right)^2 = a^2\left(1-e^2\sin^2\phi\right) \qquad \because e = \frac{c}{a} = \sqrt{1-\frac{b^2}{a^2}}$$

$$e^2 = \frac{c^2}{a^2} = \left(1-\frac{b^2}{a^2}\right)$$

$$\frac{ds}{d\phi} = a\sqrt{1-e^2\sin^2\phi}$$