Exp. Find the condition that the conics:

$ax^2+by^2=1$

$a_1x^2+b_1y^2=1$

Shall ~~cut~~ orthogonally (perpendicular).

Sol. If the two curves cut each other orthogonally, then at their point of intersection the angle between their tangents will be one right angle ($\alpha=90°$)

Given $ax^2+by^2=1$ —— (1)

$a_1x^2+b_1y^2=1$ —— (2)

We know that if the two curves cut each other orthogonally, then the condition for this is —— (3)

$$1+m_1m_2=0$$

from (1) diff. w.r.t. x

$$2ax+2by\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=m_1=-\frac{ax}{by}$$

from (II) diff. w.r.to x.

$$2a_1x+2b_1y.\frac{2dy}{dx}=0 \Rightarrow \frac{dy}{dx}=m_2=-\frac{a_1x}{b_1y}$$

from (3)

$$1+\left(\frac{-ax}{by}\right)\left(\frac{-a_1x}{+b_1y}\right)=0$$

$$aa_1x^2+bb_1y^2=0 \quad \text{—— (4)}$$

Here, x and y are the co-ordinates of the point of intersection of the two curves.

$ax^2+by^2-1=0$ —— (I)

$a_1x^2+b_1y^2-1=0$ —— (II)

from (I) − (II) $(a-a_1)x^2+(b-b_1)y^2=0$

$$(a-a_1)x^2=-(b-b_1)y^2 \quad \text{—— (5)}$$

from ④ $aa_1x^2 = -bb_1y^2$

from ⑤ $(a-a_1)x^2 = -(b-b_1)y^2$

Dividing ⑤ by ④

$$\frac{(a-a_1)\cancel{x^2}}{a\cdot a_1\cancel{x^2}} = \frac{+(b-b_1)}{+bb_1}\frac{\cancel{y^2}}{\cancel{y^2}}$$

$$\frac{a-a_1}{aa_1} = \frac{b-b_1}{bb_1}$$

$$\frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

$$\frac{1}{a_1} - \frac{1}{b_1} = \frac{1}{a} - \frac{1}{b}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$$

This is the required condition

Q.3 Find the condition that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ should intersect orthogonally.

Sol. Given curves

$$\frac{x^2}{a}+\frac{y^2}{b}=1 \quad \text{--- (i)}$$

$$\frac{x^2}{a'}+\frac{y^2}{b'}=1 \quad \text{--- (ii)}$$

Thus, Bothe curves intersect orthogonally if $1+m_1m_2=0$

$m_1=\frac{dy}{dx}$ for first curve

$m_2=\frac{dy}{dx}$ for seccond curve

Then Eqn's (i) and (ii) diff. w.r.t. x

from (i) $\frac{2x}{a}+\frac{2y}{b}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b}{a}\frac{x}{y} \Rightarrow m_1$

similarly, from (ii)

$$\frac{2x}{a'}+\frac{2y}{b'}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-b'}{a'}\frac{x}{y} \Rightarrow m_2$$

How $1+m_1m_2=0$

$$1+\left(\frac{-b}{a}\frac{x}{y}\right)\left(-\frac{b'}{a}\frac{x}{y}\right)=0$$

$$1+\frac{bb'}{aa'}\frac{x^2}{y^2}$$

$$aa'y^2+bb'x^2=0 \quad \text{--- (III)}$$

Solve the Eqn (i) & (ii)

$$\frac{x^2}{a}+\frac{y^2}{b}=1 \Rightarrow bx^2+ay^2=ab \quad \text{— (I)}$$

$$\frac{x^2}{a'}+\frac{y^2}{b'}=1 \Rightarrow b'x^2+a'y^2=a'b' \quad \text{— (II)}$$

Subtracting Eqn (I)$\times b'$ ~~and~~ (ii)$\times b$

$$\cancel{bb'x^2}+ab'y^2=abb'$$
$$\cancel{bb'x^2}+a'by^2=a'b'b$$

$$y^2(ab'-a'b)=b'(ab-a'b)$$

$$y^2=\frac{b'(ab-a'b)}{(ab'-a'b)}=\frac{bb'(a-a')}{(ab'-ba')}$$

Putting in Eqn (I)

$$bx^2+\frac{abb'(a-a')}{(ab'-ba')}=ab$$

$$bx^2=ab-\frac{abb'(a-a')}{ab'-ba'}$$

$$bx^2=\frac{ab(ab'-ba')-abb'(a-a')}{ab'-ba'}$$

$$\cancel{b}x^2=\frac{a\cancel{b}[\cancel{ab'}-ba'-\cancel{b'a}+a'b']}{ab'-ba'}$$

$$x^2=\frac{a(a'b'-ba')}{ab'-ba'}=\frac{aa'(b'-b)}{ab'-ba'}$$

Putting the values of x^2 & y^2 in Eqn (iii)

$$aa'y^2 + bb'x^2 = 0$$

$$\frac{aa'bb'(ab-a'b)}{(ab'-a'b)} + \frac{bb'a(a'b'-ba')}{(ab'-ba')} = 0$$

$$\frac{aa'bb'(a-a') + aa'bb'(b'-b)}{(ab'-a'b)} = 0$$

$$abb'b\left[(a-a') + (b'-b)\right] = 0$$

$$a - a' + b' - b = 0$$

$$(a-b) - (a'-b') = 0$$

$$a - b = a' - b'$$

Hence.