Theorem:- The necessary and sufficient condition for the differential equation

$M(x,y)\,dx + N(x,y)\,dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ denotes the partial derivatives of M and N with respect to y and x, respectively

(i) **Condition is necessary:-**

The equation $M(x,y)\,dx + N(x,y)\,dy = 0$ will be exact if

$$M\,dx + N\,dy = du \quad \text{---} \quad (i)$$

where M, N, u are some functions of x and y.

But $$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \quad \text{---} \quad (ii)$$

$\therefore$ equating coefficients of dx and dy in (i) &

we get

$$M = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$N = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the necessary condition for exactness.

(ii) Condition is sufficient:

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then $M dx + N dy = 0$ is exact.

$\Rightarrow$ Let $\int M dx = u$, where y is supposed constant while performing integration.

Then

$$\frac{\partial}{\partial x}\left[\int M dx\right] = \frac{\partial u}{\partial x}, \text{ i.e } M = \frac{\partial u}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

or $$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

[Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$]

Integrating both sides w.r.to x (taking y as const.)

$N = \frac{\partial u}{\partial y} + f(y)$, where $f(y)$ is a function of y alone.

$$M dx + N dx = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy$$

$$= \left\{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right\} + f(y) dy$$

$$= du + f(y) dy$$

$$= d\left[u + \int f(y) dy\right]$$

which shows that $M dx + N dx = 0$ is exact

$\square$

* Equations Reducible to Exact Equations:-

Sometimes a differential equation which is not exact (i.e if condition $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$); then it can be made so on multiplication by a suitable factor called an integrating factor (I.F) of two variables.

$$f(x,y)[M(x,y)\,dx + N(x,y)\,dy] = 0$$

then I.F is $f(x,y)$.

The rules for finding integrating factors of the equations $M_{(x,y)}dx + N_{(x,y)}dy = 0$

(i) If $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous differential equation with $Mx + Ny \neq 0$, then integrating factor will be $\frac{1}{Mx+Ny}$, $(Mx+Ny \neq 0)$

(ii) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is only function of x say $f(x)$, then $e^{\int f(x)dx}$ will be an integrating factor.

(iii) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is only function of y say $g(y)$ then $e^{\int g(y)dy}$ will be an integrating factor.

(iv) If given differential equation is of the form $f_1(x,y)\,y\,dx + f_2(x,y)\,x\,dy = 0$, then integrating factor will be $\frac{1}{Mx-Ny}$, where $Mx - Ny \neq 0$.

Exp. Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ (I)

This equation is homogeneous in x and y.

$M(x,y)dx + N(x,y)dy = 0$

$\Rightarrow M(x,y) = x^2y - 2xy^2 = M$

$N(x,y) = -(x^3 - 3x^2y) = N$

Thus

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y}$$

$$I.F = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Then multiplying throughout by $\frac{1}{x^2y^2}$ in Eq (I)

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{---(II)}$$

Again compare $Mdx + Ndy$

$M = \left(\frac{1}{y} - \frac{2}{x}\right) \Rightarrow \frac{\partial M}{\partial y} = \left(-\frac{1}{y^2}\right) = -\frac{1}{y^2}$

$N = \left(\frac{3}{y} - \frac{x^2}{y^2}\right) \Rightarrow \frac{\partial N}{\partial x} = -\frac{1}{y^2}$

Hence This is exact differen eqn (II)

The solution is

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy + c$$

(y const. — (term of N not containing x)

$$= \frac{x}{y} - 2\log x + 3\log y = c$$