Theorem - Any two bases of a finite dimensional vector space (FDVS) have the same number of vectors (elements).

or.

Every basis for a finite dimensional vector space must contain the same number of vectors.

Proof:- V is a FDVS and it has a basis.

Let $S = \{v_1, v_2, \dots, v_n\}$ and $T = \{w_1, w_2, \dots, w_m\}$ be any two subsets of bases of V. Then we prove that $m = n$.

(1) Suppose, $n > m$, since $v_i \in V$ and T is basis of V, there exist $\alpha_{ij} \in F$ such that $i = 1$ to n, $j = 1$ to m

$$v_i = \alpha_{i1} w_1 + \alpha_{i2} w_2 + \dots + \alpha_{im} w_m \quad \text{— (i)}$$

$$v_i = \sum_{j=1}^{m} \alpha_{ij} w_j$$

$$v_1 = \alpha_{11} w_1 + \alpha_{12} w_2 + \dots + \alpha_{1m} w_m$$

$$v_2 = \alpha_{21} w_1 + \alpha_{22} w_2 + \dots + \alpha_{2m} w_m$$

$$\vdots$$

$$v_n = \alpha_{n1} w_1 + \alpha_{n2} w_2 + \dots + \alpha_{nm} w_m$$

Let S is basis of V then S is linear independent

$$\Rightarrow \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n = 0 \quad \text{— (ii)}$$

$\beta_i \in F$, $v_i \in V$ and for $i = 1$ to n.

Substituting each v_i (for $i = 1$ to n) from Eqn (i) to in Eqn (ii), we get

$$\Rightarrow \beta_1(\alpha_{11}w_1+\alpha_{12}w_2+\cdots\alpha_{1m}w_m)+\beta_2(\alpha_{21}w_1+\alpha_{22}w_2+\cdots\alpha_{2m}w_m)$$
$$+\cdots\cdots+\beta_n(\alpha_{n1}w_1+\alpha_{n2}w_2+\cdots\alpha_{nm}w_m)=0$$

$$\therefore (\beta_1\alpha_{11}+\beta_2\alpha_{21}+\cdots\beta_n\alpha_{n1})w_1+(\beta_1\alpha_{12}+\beta_2\alpha_{22}+\cdots\beta_n\alpha_{n2})w_2$$
$$+\cdots\cdots+(\beta_1\alpha_{1m}+\beta_2\alpha_{2m}+\cdots\beta_n\alpha_{nm})w_m=0 \quad \text{—(iii)}$$

Since $w_1, w_2, \cdots w_m$ vectors are basis of T
Then these vectors are L.I i.e $w_j \neq 0$ but all scalars are zero for $j = 1$ to m. i.e

$$\left.\begin{aligned}
&\beta_1\alpha_{11}+\beta_2\alpha_{21}+\cdots+\beta_n\alpha_{n1}=0\\
&\beta_1\alpha_{12}+\beta_2\alpha_{22}+\cdots+\beta_n\alpha_{n2}=0\\
&\vdots\\
&\beta_1\alpha_{1m}+\beta_2\alpha_{2m}+\cdots\beta_n\alpha_{nm}=0
\end{aligned}\right\}$$

This system is homogeneous Linear Eqns.
It has m linear eqn and n variables.
Therefore L. Eqn (m) < variables (n) according to our consideration as $(n>m)$. Then this system has non-zero solution i.e non-trivial solution. It means $\beta_1, \beta_2, \cdots \beta_n$ in F are not all zero. Therefore $v_1, v_2, \cdots v_n$ are Linear dependent (L.D.). Which is a contradiction for $(n>m)$. Thus ous assumption $(n>m)$ is worng and so $m \geq n$ for S is L.I.

$\Rightarrow$ S. is L.I $\Rightarrow$ $n \leq m$ or $m \geq n$ — (A)

Similarly, suppose $m > n$, Since $w_j \in V$ and S is basis of V then there exist $\alpha_{ij} \in F$ for $i = 1$ to n, $j = 1$ to m such that

$$w_j = \alpha_{1j} v_1 + \alpha_{2j} v_2 + \cdots \alpha_{nj} v_n \quad \text{--- (iv)}$$

or $w_j = \sum_{i=1}^{n} \alpha_{ij} v_i$

$$w_1 = \alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{n1} v_n$$
$$w_2 = \alpha_{12} v_1 + \alpha_{22} v_2 + \cdots + \alpha_{n2} v_n$$
$$w_3 = \alpha_{13} v_1 + \alpha_{23} v_2 + \cdots \alpha_{n3} v_n$$
$$\cdots$$
$$\cdots$$
$$w_m = \alpha_{1m} v_1 + \alpha_{2m} v_2 + \cdots \alpha_{nm} v_n$$

Since, T is basis of V then linear combination of T and T is L.I

$$\beta_1 w_1 + \beta_2 w_2 + \cdots \beta_m w_m = 0 \quad \text{--- (v)}$$

Substituting each w_j from (iv) into Eqn (v), we get

$$\beta_1(\alpha_{11} v_1 + \alpha_{21} v_2 + \cdots \alpha_{n1} v_n) + \beta_2(\alpha_{12} v_1 + \alpha_{22} v_2 + \cdots + \alpha_{n2} v_n) + \cdots + \beta_m(\alpha_{1m} v_1 + \alpha_{2m} v_2 + \cdots + \alpha_{nm} v_n) = 0$$

$$(\beta_1\alpha_{11} + \beta_2\alpha_{12} + \cdots \beta_m\alpha_{1m}) v_1 + (\beta_1\alpha_{21} + \beta_2\alpha_{22} + \cdots + \beta_m\alpha_{2m}) v_2 + \cdots (\beta_1\alpha_{n1} + \beta_2\alpha_{n2} + \cdots \beta_m\alpha_{nm}) v_n = 0$$

Since, $v_1, v_2, \cdots v_n$ vectors are basis of S. Then these vectors are L.I i.e $v_i \neq 0$ $\forall$ $i = 1$ to n but all scalars are zero for $i = 1$ to n. i.e.

$$\left.\begin{aligned}
&\beta_1\alpha_{11} + \beta_2\alpha_{12} + \cdots + \beta_m\alpha_{1m} = 0 \\
&\beta_1\alpha_{21} + \beta_2\alpha_{22} + \cdots + \beta_m\alpha_{2m} = 0 \\
&\cdots \\
&\beta_1\alpha_{n1} + \beta_2\alpha_{n2} + \cdots + \beta_m\alpha_{nm} = 0
\end{aligned}\right\}$$

This is homogeneous linear Eqn of the system. It has n lin. Eqns and m variables It means linear Eqns (n) < variables (m) according to our consideration as $(m>n)$. Therefore, this system has non-zero solution i.e non-trivial solution i.e

$\beta_1, \beta_2, \dots \beta_m$ in F are not all zero

So, $w_1, w_2, \dots w_m$ are L.D, which is a contradiction. Thus our assumption $m>n$ is wrong and so that $m \leq n$. for T is L.I.

T is L.I $\Rightarrow m \leq n$ or $n \geq m$ — (B)

from Eqn (A) & (B)

for S is L.I $\Rightarrow n \leq m$ or $m \geq n$ — (A)

T is L.I $\Rightarrow m \leq n$ or $n \geq m$ — (B)

Then Hence. $m = n$ So that two bases of V have the same number of vectors (elements)

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