Euler's Equation of motion for the perfect fluid:

Let us consider a closed surface S enclosing the non-viscous fluid moving with fluid, so that S contains the same fluid particles at any time.

Now take a point P inside S. Let ρ be the density of fluid at P and δV be the elementary volume enclosing P, $\vec{q}$ be the velocity of the fluid at P.

Therefore the momentum $\vec{M}$ of the volume V in S is given by

$$\vec{M} = \int_V \vec{q}\, \rho\, dV$$

$\therefore$ The rate of change in momentum is

$$\frac{d\vec{M}}{dt} = \int_V \frac{d\vec{q}}{dt}\rho\, dV + \int_V \vec{q}\, \frac{d}{dt}(\rho\, dV)$$

$$= \int_V \frac{d\vec{q}}{dt}\rho\, dV \qquad \left[\text{As } \rho\, dV \text{ is Constant}, \therefore \frac{d}{dt}(\rho\, dV) = 0\right] \tag{1}$$

Let $\vec{F}$ be the external forces per unit mass acting on the fluid, therefore the total force on the liquid in V i.e. body force

$$= \int_V \vec{F}\rho\, dV$$

Also if p be the pressure at a point of the surface element dS, the total force on the surface i.e. surface force

$$= -\int_S p.n\, dS,$$ where n is the unit vector along the normal.

$\therefore -\int_S p.\hat{n}\,dS = -\int_V \nabla p\,dV$ (by Gauss theorem)

$\therefore$ From Newton's 2nd law of motion,
The rate of change of momentum
= total force acting on the mass,

i.e. $\int_V \frac{d\vec{q}}{dt}\rho\,dV = \int_V \vec{F}\rho\,dV - \int_V \nabla p\,dV$

$$= \int_V (\rho\vec{F} - \nabla p)\,dV$$

or $\int_V (\rho \frac{d\vec{q}}{dt} - \rho\vec{F} + \nabla p)\,dV = 0$

But the volume V enclosed in S is arbitrary;
So we have

$$\rho\frac{d\vec{q}}{dt} - \rho\vec{F} + \nabla p = 0$$

or $\frac{d\vec{q}}{dt} - \vec{F} + \frac{1}{\rho}\nabla p = 0$

or, $\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$ ——— (2)

The operator $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{q}.\nabla)$.

$\therefore$ Equation (2) becomes

$$\frac{\partial\vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p \quad \text{——— (3)}$$

As we know that

$$\vec{\nabla}(\vec{q}.\vec{q}) = 2[\vec{q}\times \text{curl}\,\vec{q} + (\vec{q}.\vec{\nabla})\vec{q}]$$

$\therefore (\vec{q}.\vec{\nabla})\vec{q} = \frac{1}{2}\nabla q^2 - \vec{q}\times \text{curl}\,\vec{q}$

Putting this value in eqn (3) we have

$$\frac{\partial\vec{q}}{\partial t} + \nabla(\tfrac{1}{2}q^2) - \vec{q}\times \text{Curl}\,\vec{q} = F - \frac{1}{\rho}\nabla p$$

or, $\frac{\partial \vec{q}}{\partial t} - \vec{q} \times \text{curl}\, \vec{q} = F - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla q^2$ — (4)

Consider a flow for which $\vec{\nabla} \times \vec{q} \neq 0$, then $\vec{\nabla} \times \vec{q} = \xi$, is called Vorticity. If the forces are conservative, we have

$F = -\nabla \Omega$, Therefore equation (4) becomes

$$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times \xi = -\nabla\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2} q^2\right) \quad \text{— (5)} \quad [\because \text{curl}\, \vec{q} = \xi$$

Eqn (5) is required Euler's equation.

Now there arises two cases.

Case I :- When $\vec{q} \times \xi = 0$ i.e. $\vec{q}$ and $\vec{\xi}$ are parallel in which case stream line and vortex line coinside. For such motion $\vec{q}$ is called a Beltrami vector and the flow a Beltrami flow. For steady and irrotation,

$\partial \vec{q} / \partial t = 0$

$\therefore$ Eqn (5) gives

$$d\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2} q^2\right) = 0$$

$$\therefore \Omega + \int \frac{dp}{\rho} + \frac{1}{2} q^2 = \text{Constant}$$

This is known as "Bernoulli's equation for steady motion.

<u>Helmholtz vorticity equation</u> :→

Eqn (5) becomes

$$\frac{\partial \vec{q}}{\partial t} + \nabla\left(\Omega + \int \frac{dp}{\rho} + \frac{1}{2} q^2\right) = \vec{q} \times \vec{\xi}$$

Taking curl of both sides and noting that curl grad $= 0$, we get

$$\text{Curl}\, \frac{\partial \vec{q}}{\partial t} + 0 = \text{Curl}\,(\vec{q} \times \vec{\xi})$$

or, $\frac{\partial}{\partial t}(\text{curl}\, \vec{q}) = \text{curl}\,(\vec{q} \times \vec{\xi})$

or, $\frac{\partial \vec{\xi}}{\partial t} = \text{curl}\,(\vec{q} \times \vec{\xi})$

$$= \vec{q}(\vec{\nabla} \cdot \vec{\xi}) - \vec{\xi}(\vec{\nabla} \cdot \vec{q}) + (\vec{\xi} \cdot \vec{\nabla})\vec{q} - (\vec{q} \cdot \vec{\nabla})\vec{\xi} \quad \text{(6)}$$

But $\vec{\nabla}\cdot\vec{\xi} = \text{div curl}\,\vec{q} = 0$ and equation of continuity $\frac{d\rho}{dt} + \rho(\vec{\nabla}\cdot\vec{q}) = 0$ gives us,

$$(\vec{\nabla}\cdot\vec{q}) = -\frac{1}{\rho}\frac{d\rho}{dt}$$

$\therefore$ Eq^n (6) becomes

$$\frac{\partial\vec{\xi}}{\partial t} = 0 + \frac{\vec{\xi}}{\rho}\frac{d\rho}{dt} + (\vec{\nabla}\cdot\vec{\xi})\vec{q} - (\vec{q}\cdot\vec{\nabla})\vec{\xi}$$

or, $\left[\frac{\partial}{\partial t} + \vec{q}\cdot\vec{\nabla}\right]\vec{\xi} = \frac{\vec{\xi}}{\rho}\frac{d\rho}{dt} + (\vec{\xi}\cdot\vec{\nabla})\vec{q}$

or, $\frac{d\vec{\xi}}{dt} = \frac{\vec{\xi}}{\rho}\frac{d\rho}{dt} + (\vec{\xi}\cdot\vec{\nabla})\vec{\xi}$

or, $\frac{1}{\rho}\frac{d\vec{\xi}}{dt} - \frac{\vec{\xi}}{\rho^2}\cdot\frac{d\rho}{dt} = (\vec{\xi}\cdot\vec{\nabla})\frac{\vec{q}}{\rho}$

or, $\frac{d}{dt}\left(\frac{\vec{\xi}}{\rho}\right) = \left(\frac{\vec{\xi}}{\rho}\cdot\vec{\nabla}\right)\vec{q}$

This is known as "Helmholtz verticity eq^n.

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