The error of the trapezoidal formula can be obtained in the following way.

Let $y=f(x)$ be continuous and possess continuous derivatives in $[x_0, x_n]$. Expanding y in a Taylor's series around $x=x_0$ we obtain

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2}y_0'' + \cdots\right]dx \quad \text{(I)}$$

$$= y_0[x]_{x_0}^{x_1} + \left[\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_1} y_0' + \frac{1}{2}y_0''\left[\frac{(x-x_0)^3}{3}\right]_{x_0}^{x_1} \ldots$$

$$x_1 - x_0 = h$$

$$\int_{x_0}^{x_1} y\,dx = hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \cdots \quad \text{(II)}$$

Similarly

$$\frac{h}{2}(y_0+y_1) = \frac{h}{2}\left(y_0+y_0+hy_0'+\frac{h^2}{2}y_0''+\frac{h^3}{6}y_0''' \ldots\right)$$

$$\frac{1}{2}(y_0+y_1) = hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \cdots \quad \text{(III)}$$

From (III) substracting from (ii)

$$\int_{x_0}^{x_1} y\,dx - \frac{h}{2}(y_0+y_1) = -\frac{1}{12}h^3 y_0'' + \cdots$$

which is the error in the interval $[x_0, x_1]$. Proceeding in a similar manner for error in intervals $[x_1, x_2], [x_2, x_3] \ldots [x_{n-1}, x_n]$.

Thus

$$E = -\frac{1}{12}h^3\left[y_0'' + y_1'' + y_2'' \ldots y_{n-1}''\right]$$

$$E = -\frac{1}{12}h^3 n y''(\bar{x}) = -\frac{b-a}{12}h^2 y''(\bar{x})$$

Since $nh = b-a$

s total error, $y''(\bar{x})$ is the largest value of the n quantities.

Exp. Evaluating error of the trapezoidal rule for $I = \int_0^2 \frac{dx}{1+x^2}$ and $h = 0.5$

Solution. Given that

$$I = \int_0^2 \frac{1}{1+x^2}\,dx$$

$$h = 0.5 \qquad n = \frac{b-a}{h} = \frac{2.0-0}{0.5} = 4$$

Then x and y table values for Trapezoidal Rule

x	$y = \frac{1}{1+x^2}$
0	1.0
0.5	0.800
1.0	0.500
1.5	0.3076
2.0	0.200

$\bar{x} = 1.0$

Find Value Trapezoidal Rule.

$$T_n = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$T_5 = \frac{0.5}{2}\left[1.0 + 2(0.8 + 0.5 + 0.3076) + 0.2\right]$$

$$= \frac{0.5}{2}\left[1.2 + 2(1.6076)\right]$$

$$= 0.25\,(1.2 + 3.2152)$$

$$= 0.25\,[4.4152]$$

$$T_5 = 1.1038$$

The error of trapezoidal Rule

$$E^T(y) = -\frac{h^2}{12}(b-a)\,y''(\bar{x})$$

Here $b = 2, \; a = 0 \quad h = 0.5$

$b - a = 2$

We bound $|y''(x)|$ by $\max_{0 \le x \le 2} |y''(x)|$

Then

calculate derivatives.

$$y'(x) = \frac{-2x}{(1+x^2)^2}$$

$$y''(x) = \frac{-2+6x^2}{(1+x^2)^3}$$

$$y'''(x) = \frac{24(1-x^2)}{(1+x^2)^4} = 0 \text{ for } x = 1$$

Then

$$y''(0) = \frac{-2+6\times(0)^2}{(1+0^2)^3} = \frac{-2-0}{1+0} = -2$$

$$y''(1) = \frac{-2+6\times(1)^2}{(1+(1)^2)^3} = \frac{-2+6}{8} = \frac{4}{8} = \frac{1}{2} = 0.5$$

$$y''(2) = \frac{-2+6\times(2)^2}{(1+(2)^2)^3} = \frac{-2+24}{125} = \frac{22}{125} = 0.176$$

$$\max_{0 \le x \le 2} \text{(largest)} |y''(x)| = \max\{|y''(0)|, |y''(1)|, |y''(2)|\} = 2$$

$$= \max\{|-2|, |0.5|, |0.176|\} = 2$$

Then.

$$E^T(y) = -\frac{(0.5)^2}{12} \times 2 \times 2$$

$$= -0.08333$$

Exp. Use Trapezoidal Rule to compute $\int_1^2 \frac{1}{x} dx$ using three intervals compare it with exact Value.

Sol. Give that $\int_{a=1}^{b=2} \frac{1}{x} dx$

$n = 3, \ a = 1, \ b = 2 \qquad h = \frac{b-a}{n} = \frac{2-1}{3} = 1/3$

$y = \frac{1}{x} \qquad y = 1/x$

$x_0 = 1 \qquad y_0 = \frac{1}{1} = 1$

$x_0 + h = x_1 = 1 + 1/3 = 4/3, \quad y_1 = \frac{1}{4/3} = 3/4$

$x_1 + h = x_2 = 4/3 + \frac{1}{3} = 5/3, \quad y_2 = \frac{1}{5/3} \ 3/5$

$x_2 + h = x_3 = \frac{5}{3} + \frac{1}{3} = 6/3 = 2, \quad y_3 = 1/2$

$$T_3 = \frac{h}{2}[y_0 + 2(y_1 + y_2) + y_3]$$

$$= \frac{1}{6}\left[1 + 2\left(\frac{3}{4} + \frac{3}{5}\right) + \frac{1}{2}\right]$$

$$= \frac{1}{6}\left[1.5 + 2\left(\frac{15+12}{20}\right)\right]$$

$$= \frac{1}{6}\left[1.5 + \frac{27}{10}\right]$$

$$= \frac{1}{6}[1.5 + 2.7]$$

$$\int_1^2 \frac{1}{x} dx = \frac{4.2}{6} = 0.70$$

By Direct integrating then

$$\int_1^2 \frac{1}{x} dx = [\log x]_1^2 = \log 2 - \log 1$$

$$= 0.693 - 0$$

$$= 0.693$$

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