ELASTICITY

The internal property of a material body by virtue of which it regain its original condition on removal of the deforming force is called elasticity.

Hooke's Law: It states that "within elastic limit the stress is proportional to strain. i.e. Stress α Strain or Stress/Strain = E

Corresponding to three types of strains have three kinds of elastic constants as

Young's Modulus of elasticity (Y)

$$Y = \frac{\text{applied load per unit area of cross section}}{\text{change in length per unit length}}$$

$$= \frac{F/A}{\Delta l/L} = \frac{F}{A}\frac{L}{\Delta L}$$

Bulk Modulus of elasticity (K) $= \frac{\text{Stress}}{\text{volume Strain}}$

$$= \frac{P/A}{\Delta V/V} = \frac{P}{A}\frac{V}{\Delta V}$$

Modulus of Rigidity (η) $= \frac{\text{Tangential Strain}}{\text{shearing Strain}}$

$$= \frac{F}{A.\theta}$$

Besides these there are also some constants - as

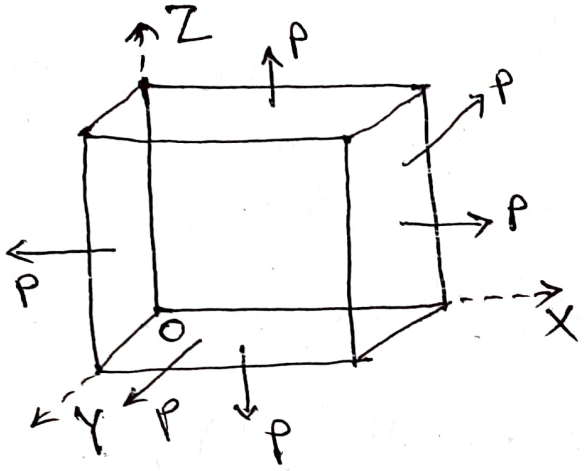
Axial Modulus (X) is defined interms of principal stress needed to produce a simple elongation without lateral change. The complete stress is a Young's modulus stress together with two perpendicular stress of such a magnitude as will prevent lateral contraction.

Thus $X = \frac{\text{Longitudinal load per unit area of cross-section}}{\text{Increase in length per unit length}}$

Poisson's Ratio(σ): When a body undergoes a linear tensile strain, it experiences a lateral contraction in addition to its longitudinal extension. This lateral strain is proportional to

longitudinal strain. Thus ratio of lateral / longitudinal strain is a constant called Poisson's ratio (σ).

Relation between Elastic Constants Y, K, η, σ and X



To obtain the relation, let us consider a unit cube, as shown in figure, with one of it corners O as the origin of rectangular co-ordinates. Let the cube be subjected to equal sets of deforming forces acting along the coordinate axes.

obviously a set of forces (P-P) acting along the OX direction will produce longitudinal extensional strain along OX of value $+P/Y$ and contractional or lateral strain of magnitude $-\sigma P/Y$ along the other two axes OY and OZ respectively. Similar will be the effects of other set of forces.

Let us record the changes caused by each of them respectively in the tabular form.

— Stress —			— Strain —		
OX	OY	OZ	OX	OY	OZ
$+P$	0	0	$+\frac{P}{Y}$	$-\sigma P/Y$	$-\sigma P/Y$
0	$+P$	0	$-\sigma P/Y$	$+P/Y$	$-\sigma P/Y$
0	0	$+P$	$-\sigma P/Y$	$-\sigma P/Y$	$+P/Y$
$+P$	$+P$	$+P$	$+\frac{P}{Y}(1-2\sigma)$	$+\frac{P}{Y}(1-2\sigma)$	$+\frac{P}{Y}(1-2\sigma)$ → Sum

In this way, the table shows the result of applying three perpendicular stresses, each $+P$, in succession. The resultant stress is a pure volume stress $+P$ and volume strain is three perpendicular extensions each equal to $P/Y(1-2\sigma)$, or a volume strain of $\frac{3P}{Y}(1-2\sigma)$. But by definition volume strain is P/K and thus

$$\frac{P}{K} = \frac{3P}{Y}(1-2\sigma) \quad \text{or} \quad \boxed{Y = 3K(1-2\sigma)} \quad -(1)$$

Relation between γ, η and σ

Let the same unit cube be subjected to a tensile stress $(p-p)$ along x-axis and a compressive stress $(p-p)$ along y and z-axis. These elongation and contraction together may be looked upon as equivalent to pure shear in x-y plane. The results are shown in tabular form.

	Stress			Strain		
	σx	σy	σz	σx	σy	σz
	$+p$	0	0	$+p/\gamma$	$-\sigma p/\gamma$	$-\sigma p/\gamma$
	0	$-p$	0	$+\sigma p/\gamma$	$-p/\gamma$	$+\sigma p/\gamma$
Sum	$+p$	$-p$	0	$\frac{p}{\gamma}(1+\sigma)$	$-\frac{p}{\gamma}(1+\sigma)$	0

Thus the equivalent shear produced is

$$\theta = \frac{2p}{\gamma}(1+\sigma)$$ and stress is p.

If η denotes the modulus of rigidity, then

$$\eta = \frac{p}{\theta} = \frac{p}{\frac{2p}{\gamma}(1+\sigma)} = \frac{\gamma}{2(1+\sigma)}$$

$$\therefore \boxed{\gamma = 2\eta(1+\sigma)} \quad \text{---} \quad (2)$$

Relation between γ, η and σ

Let us consider the same cube be subjected to tensile stress p on the two opposite faces perpendicular to x-axis and tensile stresses p_1 on the other four faces, the relation between p and p_1 being such as to produce a resultant elongation along x-axis only a no net change of dimension along y or z-axis.

	Stress			Strain		
	σx	σy	σz	σx	σy	σz
	$+p$	0	0	$+p/\gamma$	$-\sigma p/\gamma$	$-\sigma p/\gamma$
	0	$+p_1$	0	$-\sigma p_1/\gamma$	$+p_1/\gamma$	$-\sigma p_1/\gamma$
	0	0	$+p_1$	$-\sigma p_1/\gamma$	$-\sigma p_1/\gamma$	$+p_1/\gamma$
Sum	$+p$	$+p_1$	$+p_1$	$\frac{p}{\gamma} - 2\sigma p_1/\gamma$	$\frac{p_1}{\gamma}(1-\sigma) - \sigma p/\gamma$	$+p_1/\gamma(1-\sigma) - \frac{\sigma p}{\gamma}$

In order that the cube undergoes no change along OY and OZ directions we must have

$$\frac{p_1}{Y}(1-\sigma) - \frac{p}{Y}\sigma = \frac{1}{Y}\{p_1 - \sigma(p+p_1)\} = 0$$

or $p_1 - \sigma(p+p_1) = 0$

or $p_1 = \frac{\sigma p}{(1-\sigma)}$

In this case the extension along x-axis is

$$\frac{p}{Y} - 2\sigma\frac{p_1}{Y} = \frac{p}{Y} - \frac{2\sigma^2 p}{(1-\sigma)Y}$$ using the value of p_1

$$= \frac{p}{Y}\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}$$

But, by definition, the axial modulus

$$\chi = \frac{p}{\text{strain along x-axis}}$$

$$= \frac{p}{\frac{p}{Y}\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}} = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)}$$

$$\therefore \quad Y = \frac{\chi(1+\sigma)(1-2\sigma)}{(1-\sigma)} \quad \text{— (3)}$$

Now from equation (1) and (2)

$3K(1-2\sigma) = 2\eta(1+\sigma)$

or $3K - 2\eta = 2\sigma(\eta + 3K)$

$$\therefore \boxed{\sigma = \frac{3K-2\eta}{2\eta+6K}} \quad \text{— (4)}$$

Again from (2), $\sigma = \frac{Y}{2\eta} - 1$

and using this value in eq^n (1)

$$Y = 3K\left(3 - \frac{Y}{\eta}\right) = \frac{9\eta K - 3KY}{\eta}$$

or $$\boxed{Y = \frac{9\eta K}{\eta + 3K}} \quad \text{— (5)}$$

Again from equation (4)

$$1+\sigma = 1+\frac{3K-2\eta}{6K+2\eta} = \frac{9K}{6K+2\eta}$$

$$1-\sigma = \frac{3K+4\eta}{6K+2\eta}$$

and $$1-2\sigma = \frac{3\eta}{3K+\eta}$$

Using these relations in eqn (3)

$$\gamma = \frac{3K+4\eta}{3} \quad \text{——(6)}$$

Equations (1) to (6) give the interrelations between different elastic constants.

Limiting values of σ

From eqn (1) and (2)

$$3K(1-2\sigma) = 2\eta(1+\sigma)$$

Where K and η are essentially positive

$$\therefore \frac{1+\sigma}{1-2\sigma} = \frac{3K}{2\eta} = \text{a +ve quantity}$$

But when $\sigma < -1$, $\frac{1+\sigma}{1-2\sigma}$ becomes -ve. Also when $\sigma > \frac{1}{2}$, $\frac{1+\sigma}{1-2\sigma}$ is negative. But as this quantity cannot be negative, σ cannot be less than -1 or greater than $+\frac{1}{2}$. Hence σ lies between -1 and $\frac{1}{2}$.

It is to be noted that negative value of σ would mean that, on being extended, a body would also expand laterally. Again the value of $\sigma = \frac{1}{2}$ would mean that the substance is perfectly incompressible.

Experiments show that the value of σ lies between 0.2 and 0.4 for most materials.