Show that Dirac equation is covariant under Lorentz transformation.

Let us first see what is meant by Covariance under Lorentz transformation. Consider two observers associated with Lorentz frames Σ and Σ'. We require that scalar quantities must remain the same in the two systems while four vectors must transform like space-time coordinates. So we require that relativistic wave equation for electron in primed system look like Dirac equation and there should be a explicit prescription that relates $\psi(x_\mu$ and $\psi'(x'_\mu))$; where $\psi(x_\mu)$ and $\psi'(x'_\mu)$ are the wave functions corresponding to a given physical situation as viewed from unprimed and primed system using the above prescription; we should be able to show that relativistic wave equation for electron in primed co-ordinate system is actually equivalent to the Dirac equation.

Here gamma matrices are introduced merely as useful short hand devices that rearrange the components of ψ. Therefore, gamma matrices can be assumed to be unchanged under Lorentz transformation. Gamma matrices themselves are not to be considered as the components of a four vector even though $\bar{\psi}\gamma_\mu\psi$ does transform like a four vector.

The equations in Σ and Σ' are to be

$$\left[\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right]\psi(x_\mu) = 0 \quad \text{--- (1)}$$

$$\left[\gamma_\mu \frac{\partial}{\partial x_\mu'} + \frac{mc}{\hbar}\right]\psi'(x_\mu') = 0 \quad \text{--- (2)}$$

x_μ and $x_{\mu'}$ are related by

$$x_\mu = a_{\mu\nu} x_\mu'$$

$$\frac{\partial\psi}{\partial x_\mu'} = \frac{\partial\psi}{\partial x_\nu} \cdot \frac{\partial x_\nu}{\partial x_\mu'} = a_{\mu\nu}\frac{\partial\psi}{\partial x_\nu}$$

In analogy to the transformation of a four-vector, we expect that $\psi(x_\mu)$ and $\psi'(x_\mu')$ be related by a linear transformation.

Then

$$\psi'(x_\mu') = S\psi(x_\mu) \quad \text{--- (3)}$$

Where S is a 4×4 matrix which depends only on the nature of Lorentz transformation and is completely independent of $\vec{r}$ and t.

Eqn. (2) may now be written as —

$$\left[\gamma_\mu a_{\mu\nu}\frac{\partial}{\partial x_\nu} + \frac{mc}{\hbar}\right]S\psi(x_\mu) = 0$$

Multiplying from the left by S^{-1}; we get

$$\left[S^{-1}\gamma_\mu\, a_{\mu\nu}\, S\frac{\partial}{\partial x_\nu} + \frac{mc}{\hbar}\right]\psi(x_\mu) = 0 \qquad (4)$$

Equation (4) is equivalent to the Dirac equation (1) provided we can find a matrix S that satisfies

$$S^{-1}\gamma_\mu\, a_{\mu\nu}\, S = \gamma_\nu$$

Since S rearranges the components of ψ under Lorentz transformation, whereas $a_{\mu\nu}$ rearranges the components of a four vector; therefore, these two matrices commute.

So

$$S^{-1}\gamma_\mu\, S\, a_{\mu\nu} = \gamma_\nu$$

multiplying $a_{\lambda\nu}$ and summing over ν, we obtain

$$\sum_\nu (S^{-1}\gamma_\mu S)\, a_{\mu\nu}\, a_{\lambda\nu} = \gamma_\nu\, a_{\lambda\nu}$$

or $$\sum_\nu (S^{-1}\gamma_\mu S)\,\delta_{\mu\lambda} = \gamma_\nu\, a_{\lambda\nu}$$

or $$S^{-1}\gamma_\lambda\, S = \gamma_\nu\, a_{\lambda\nu} \qquad (5)$$

The problem of demonstrating the relativistic covariance of Dirac equation is now reduced to that of finding an S that satisfies eqn (5).