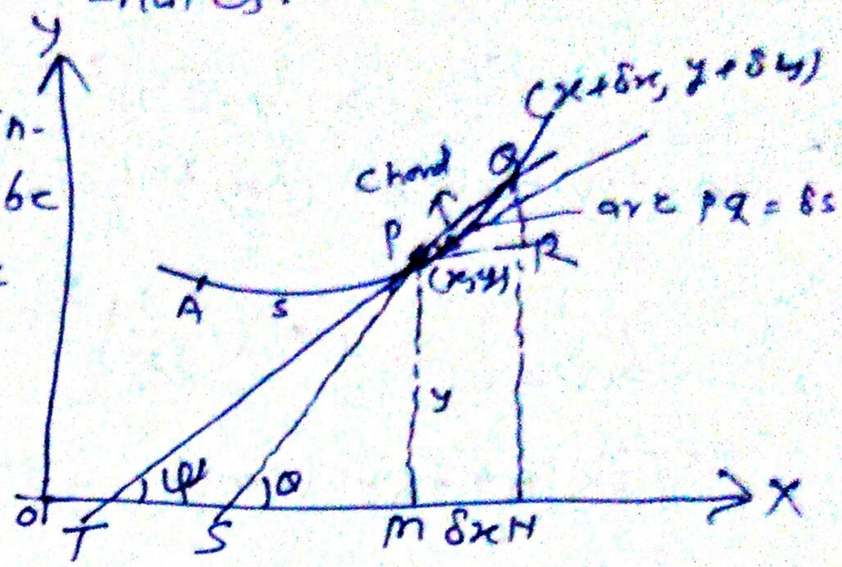
Derivatives of arc- length of Cartesion Co-ordinates.

(आर्क) (Kartizian)

Let P be any point on the curve whose co-ordinates are (x, y). Let Q be any other point on the curve which is very near to P.



The co-ordinates of Q are $(x+\delta x, y+\delta y)$.

PM & QN are $\perp^{lar}$ (perpendicular) on x-axis from P and Q respectively. Then.

$OM = x$ & $PM = y$

$ON = x + \delta x$ & $QN = y + \delta y$

From P draw PR $\perp^{lar}$ to QN. $\because PM = RN = y$

$MN = ON - OM = x + \delta x - x = \delta x$

$QR = QN - RN = y + \delta y - y = \delta y$

Let A be any point on the curve where arc AP = s

arc AQ = arc AP + arc PQ

$AQ = s + \delta s$

In Right triangled triangle (△) PRQ

$$(PQ)^2 = PR^2 + QR^2 \quad \text{——} (1)$$

But we know that when $Q \to P$, then

$$\frac{\text{chord } PQ}{\text{arc } PQ} \to 1, \quad \because \text{chord } PQ \to \text{arc } PQ$$

∴ In the limimiting position, we can write arc PQ instead of chord PQ

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 \quad \text{—— (ii)}$$

on divided by $(\delta x)^2$ both sides

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

Taking the limit $\delta x \to 0$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Again from (ii)

$$\left(\frac{\delta s}{\delta y}\right)^2 = \left(\frac{\delta x}{\delta y}\right)^2 + 1$$

∴ Taking limit.

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$$

$$\Rightarrow \left(\frac{ds}{dy}\right) = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Again from (ii) divided by $(\delta t)^2$ both sides

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

Taking limit.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Angle between the tangent and x-axis

Let the curve $y = f(x)$ and point $P(x, y)$ on the curve.

From P, a tangent PT meets OX in point T and make angle φ at T. $\angle PTX = \varphi$

QP extends, which meets the x-axis at point S and makes angle θ at S. i.e $\angle PS\cancel{X} = \theta$

$\because \angle QPR = \theta$

In ΔQPR

$$\sin\theta = \frac{QR}{PQ} \Rightarrow \sin\theta = \frac{\delta y}{PQ} \quad \text{---(i)}$$

$$\cos\theta = \frac{PR}{PQ} \Rightarrow \sin\theta = \frac{\delta x}{PQ} \quad \text{---(ii)}$$

$$\tan\theta = \frac{QR}{PR} \Rightarrow \tan\theta = \frac{\delta y}{\delta x} \quad \text{---(iii)}$$

Let $Q \to P$ then chord PQ becomes the tangent at P.

Hence $\theta \to \varphi$, and $\delta x \to 0$

then $\dfrac{\text{Chord } PQ}{\text{arc } PQ} \to 1 \qquad \because$ chord $PQ = \delta s$

Then in the limiting position when $Q \to P$ from (i), (ii) and (iii)

$\theta \to \varphi$

$$\sin\theta = \frac{dy}{ds} \Rightarrow \sin\varphi = \frac{dy}{ds}$$

$$\cos\theta = \frac{dx}{ds} \Rightarrow \cos\varphi = \frac{dx}{ds}$$

$$\tan\theta = \frac{dy}{dx} \Rightarrow \tan\varphi = \frac{dy}{dx}$$

(I) If the tangent is $\parallel$ to x-axis then $\varphi = 0$ $\because \frac{dy}{dx} = 0$

(II) If the tangent is $\parallel$ to y-axis then $\varphi = 90^\circ$

$\because \dfrac{dy}{dx} = \infty$ or $\dfrac{dx}{dy} = 0$

Exp Find $\frac{ds}{dx}$ from the curve $x^{2/3}+y^{2/3}=a^{2/3}$

Sol. Given curve

$$x^{2/3}+y^{2/3}=a^{2/3} \quad \text{—— (1)}$$

Diff. w.r.t. x, we get

$$\frac{2}{3}\cdot x^{\frac{2}{3}-1}+\frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx}=0$$

$$\frac{2}{3}\cdot x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

Now $\left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$

$$=1+\left(\frac{-y^{1/3}}{x^{1/3}}\right)^2$$

$$=1+\frac{y^{2/3}}{x^{2/3}}$$

$$\left(\frac{ds}{dx}\right)^2=\frac{x^{2/3}+y^{2/3}}{x^{2/3}}=\frac{a^{2/3}}{x^{2/3}}$$

$$\left(\frac{ds}{dx}\right)^2=\left(\frac{a^{1/3}}{x^{1/3}}\right)^2$$

$$\frac{ds}{dx}=\frac{a^{1/3}}{x^{1/3}}=\left(\frac{a}{x}\right)^{1/3}$$

—x—