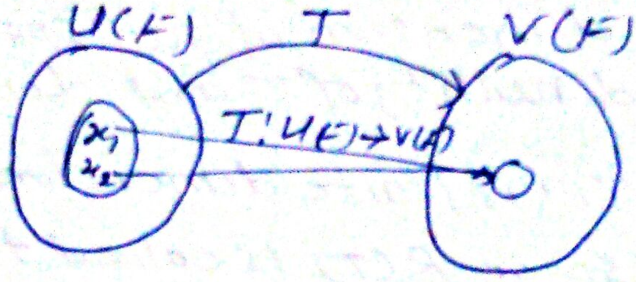
* Kernel of a Homomorphism (L.T):

Let T be a homomorphism of a vector space U into a vector space V over field (F) (i.e $T: U \to V$). Then the set K of all the vectors/elements of U which are mapped into zero (0) element of V is called kernel of homomorphism and is defined as

$$K = \{x \in U : T(x) = 0, \text{ where } 0 \text{ is the zero vector of } V\}$$



and denoted by $\ker T$ or $N(T)$

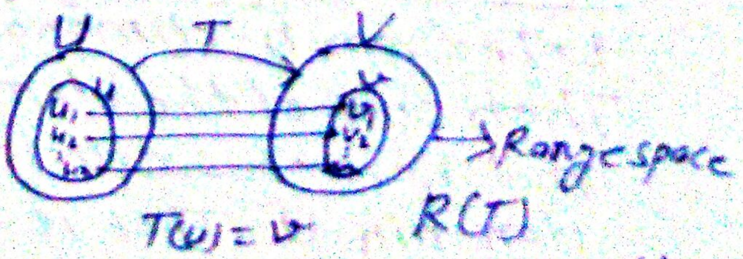
$\ker T$ is also called the null space of T

Def: Let $T: U \to V$ be L.T. Then the set of all vectors u in U that satisfy $T(u) = 0$ is called $\ker T$.

* Range and Null Space of a Linear Transformation

Let U and V be two vector spaces and T be a linear transformation from U into V i.e $T: U \to V$. Then the image or range of T denoted as $R(T)$ is the set of all vectors $v \in V$ such that

$$R(T) = \{v \in V : T(u) = v, \text{ for some } u \in U\}$$



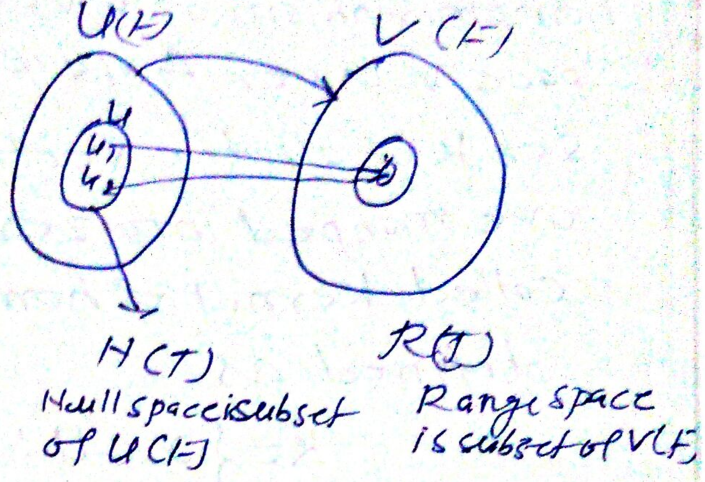
and null space of T denoted as $N(T)$ is the set of all vector $u \in U$ such that $T(u) = 0$ i.e image of u is zero.

$N(T) = \{u \in U : T(u) = 0 \in V\}$

(i) $\ker(T) \subset U(F)$

(ii) $N(T) \subset U(F)$

(iii) $R(T) \subset V(F)$



→ If $U(F)$ is finite dimensional, then then dimension of nullspace $N(T)$ is called nullity of T and denoted $\eta(T)$

→ If $V(F)$ is finite dimensional, then the range space $R(T)$ is called the rank of T and denoted by $\rho(T)$.

Rank and Nullity of a Linear Transformation

Let $T: U(F) \to V(F)$ be a linear transformation where $U(F)$ is finite dimensional vector space. The dimension of the kernel of T is called nullity of T and denoted by $\eta(T)$. The dimension of the range of T is called the rank of T and is denoted by $\rho(T)$.

Sum of Rank and Nullity:- Let $T: U \to V$ be a L.T. from n-dimenal vector space U into V. Then sum of the dimensions of the range and kernel is equal to the dimension of the domain U. That is

$$\rho(T) + \eta(T) = n$$

or $$\dim(R(T)) + \dim(\ker T) = \dim(U)$$

Theorem:- Let U and V be vector spaces over a field F and let T be a linear Transformation from U into V i.e $T: U \to V$. If U is finite-dimensional then

$$\dim(U) = \text{rank}(T) + \text{nullity}(T)$$

$$\text{or } \dim U = \rho(T) + \eta(T)$$

Proof Let $\dim U = n$ and $\dim N(T) = m$ i.e $\eta(T) = m$ or $\dim \ker(T) = m$

Let $S_1 = \{u_1, u_2, \cdots u_m\}$ be basis of $\ker(T) \subseteq U$

Since $u_i \in \ker(T)$ for $i = 1, 2, \cdots m$

$\therefore T(u_i) = 0$ ——— (i)

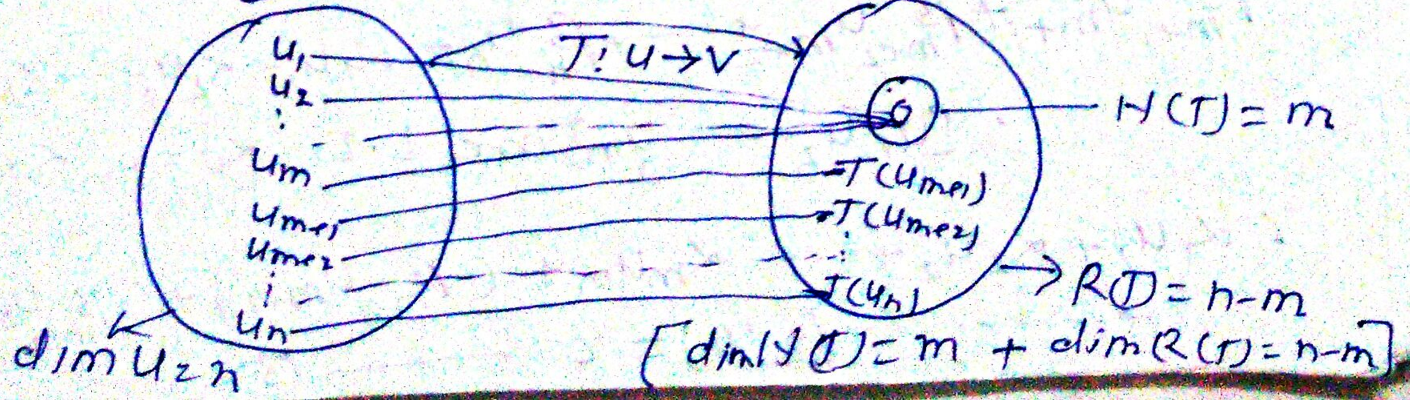
Therefore, $S_1 = \{u_1, u_2 \cdots u_m\}$ are linear independent (L.I.) vectors in U. They can be extendended tor form a basis of U i.e there exist vectors-

$u_{m+1}, u_{m+2} \cdots u_n$ in U.

$\therefore S_2 = \{u_1, u_2 \cdots u_m, u_{m+1}, \cdots u_n\}$ is a basis of U.

So, we show that

$S_3 = \{T(u_{m+1}), T(u_{m+2}), \cdots T(u_n)\}$ is a basis of $T(U)$



Let $v \in T(U)$ be arbitary. Then there exist som $u \in U$ such that $v = T(u)$

Since S_2 is basis of U vector space then

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + \alpha_{m+1} u_{m+1} + \cdots \alpha_n u_n$$

$$\alpha_i \in F \text{ (scalars) for } i = 1, \cdots n$$

$$\therefore T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + \alpha_{m+1} u_{m+1} + \cdots \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots \alpha_m T(u_m) + \alpha_{m+1} T(u_{m+1}) + \cdots \alpha_n T(u_n)$$ as T is L.T.

$$\therefore T(u) = 0 + \alpha_{m+1} T(u_{m+1}) + \alpha_{m+2} T(u_{m+2}) + \cdots \alpha_n T(u_n)$$

This prove that $T(U) = L(S_3)$ — using ① $T(u_i) = 0$ for $i = 1, 2 \cdots m$

Now we show that S_3 is L. I. subset of U.

Let

$$\beta_{m+1} T(u_{m+1}) + \beta_{m+2} T(u_{m+2}) + \cdots + \beta_n T(u_n) = 0$$

$$T(\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n) = 0$$

$$\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n = 0 \quad \text{—— ②}$$

As T is L.T.

$$\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m$$

$\because \{u_1, u_2, \cdots u_m\}$ are L.I $\alpha_i = 0$ $\forall i$

$$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + (-\beta_{m+1} u_{m+1}) + (-\beta_{m+2} u_{m+2}) + \cdots (-\beta_n u_n) = 0 \quad \text{—— ③}$$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 \cdots \alpha_m = 0,\ \beta_{m+1} = \beta_{m+2} = 0,$

$\beta_{m+3} = \beta_{m+4} \cdots\cdots \beta_n = 0$

Since S_2 is a linear Independent.
from ② and (3), it follow that S_3 is L.I. subset of $T(U)$.
Thus S_3 is basis of $T(U)$ as $(n-m)$

$$\dim T(U) = \dim(U) - \dim \ker(T)$$

$$\dim(U) = \dim T(U) + \dim N(T)$$

$$\dim(U) = \text{rank}(T) + \text{nullity}(T)$$

$$\dim U = \rho(T) + \eta(T)$$

Hence proved