

Ex. Solve $x^2(y - px) = yp^2$

Sol. Given equation:

$$x^2(y - px) = yp^2 \quad \text{--- (i)}$$

is not Clairaut's differential equation form

Therefore, taking $x^2 = u$ and $y^2 = v$ then we can convert it into the Clairaut's form.

$$x^2 = u \Rightarrow 2x dx = du$$

$$y^2 = v \Rightarrow 2y dy = dv$$

$$\therefore \frac{dv}{dx} = \frac{2y dy}{2x dx} = \frac{y}{x} \frac{dy}{dx} = \frac{y}{x} p \Rightarrow p = \frac{x}{y} \frac{dv}{du}$$

These are putting in Eqn (i), then given eqn. reduces to

$$x^2 \left(y - \frac{x^2}{y} \frac{dv}{du} \right) = y \cdot \frac{x^2}{y^2} \left(\frac{dv}{du} \right)^2$$

$$y^2 - x^2 \frac{dv}{du} = \left(\frac{dv}{du} \right)^2$$

$$v - u \frac{dv}{du} = \left(\frac{dv}{du} \right)^2$$

$$v = u \frac{dv}{du} + \left(\frac{dv}{du} \right)^2 \quad \text{--- (ii)}$$

Which is in Clairaut's differential equation form. Hence, the general solution Eqn (ii) can be obtained by replacing $\frac{dv}{du} = c$

$$\text{Then- } v = uc + c^2$$

$$\text{and } y^2 = x^2 c + c^2$$

where c is arbitrary

constant. This is the general solution.

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Exp. Solve $\sin px \cos y = \cos px \sin y + p$

Sol. The given equation is not of the Clairaut's form.
Thus.

$$\sin px \cos y - \cos px \sin y = p \quad \text{--- (i)}$$

$$\sin(px - y) = p$$

$$px - y = \sin^{-1} p$$

$$y = px - \sin^{-1} p \quad \text{--- (ii)}$$

Which is in Clairaut's diff. Eqn. form.

So, p replacing by c then find general solution of Eqn (ii)

$$y = cx - \sin^{-1} c \quad \& \quad x = \frac{1}{\sqrt{1-p^2}} \Rightarrow p = \frac{\sqrt{x^2}}{x}$$

From (i) $y = \sqrt{x^2 - 1} - \sin^{-1} \frac{\sqrt{x^2}}{x}$

Exp. Solve $e^{4x} (p-1) + e^{2y} p^2 = 0 \quad \text{--- (1)}$

Sol. The given equation is not in Clairaut's form, to convert it equation into the Clairaut's form by taking $e^{2x} = u$ and $e^{2y} = v$

$$\text{then } e^{2x} = u \Rightarrow 2 \cdot e^{2x} dx = du$$

$$e^{2y} = v \Rightarrow 2 \cdot e^{2y} dy = dv$$

$$\frac{dv}{du} = \frac{e^{2y} dy}{e^{2x} dx} = \frac{e^{2y}}{e^{2x}} p \Rightarrow p = \frac{e^{2x}}{e^{2y}} \frac{dv}{du}$$

$$\text{From Eqn (1)} \quad e^{4x} \left(\frac{e^{2x}}{e^{2y}} \frac{dv}{du} - 1 \right) + e^{2y} \cdot \frac{e^{4x}}{e^{4y}} \left(\frac{dv}{du} \right)^2 = 0$$

$$\frac{e^{2x}}{e^{2y}} \frac{dv}{du} - e^{2y} + \left(\frac{dv}{du} \right)^2 = 0$$

$$v = u \frac{dv}{du} + \left(\frac{dv}{du} \right)^2 \quad \text{is in Clairaut's form}$$

$$\text{The Solution is } \Rightarrow v = uc + c^2 \Rightarrow e = e^{2x} c + c^2$$

* Orthogonal Trajectories

In first-order differential equations with variables separable, Orthogonal (perpendicular) trajectories are curves that intersect the lines of the given planar family $\phi(x, y, a) = 0$, (where a is a parameter) at a right angle (perpendicular $\perp$ angle).

If $F(x, y, y') = 0$ is the differential equation of the family, then

$$F(x, y, -\frac{1}{y'}) = 0$$

is the differential equation of the orthogonal trajectories.

Exp. Find the orthogonal trajectories of the family of ellipses $\rightarrow x^2 + 2y^2 = a^2$

Sol. Given equation

$$x^2 + 2y^2 = a^2 \quad (1)$$

Differentiating its equation, w.r.to x .

$$2x + 2 \cdot 2yy' = 0$$

$$x + 2yy' = 0 \quad (ii)$$

Whence, replacing $y' = -\frac{1}{y}$, we get

$$x - 2y \cdot \frac{1}{y} = 0 \quad y' = \frac{2y}{x}$$

Integrating, we have $\int \frac{dy}{y} = 2 \int \frac{dx}{x} + c$

$$\Rightarrow y = c \underline{x^2} \quad \int \frac{1}{y} = 2 \log y = 2 \log x + c$$