Gravitational Field and Potential

It is quite evident from Newton's law of gravitation that everybody attracts another body. Naturally, the space surrounding the body up to where, the force of attraction effectively works, is called gravitational field.

It is measured by a term called <u>Intensity of Gravitational Field</u>. It is defined as the force experienced by unit test placed at that point.

ie $\vec{E} = \frac{\vec{F}}{m_0}$ where $\vec{F}$ is the force
m_0 is the test mass
& $\vec{E}$ = Intensity

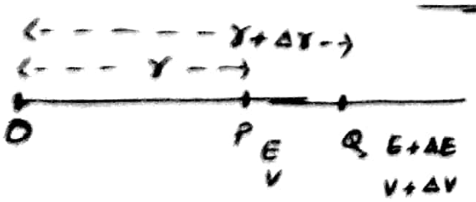
If $m_0 = 1$, the $\vec{E} = \vec{F}$

Gravitational Potential

The gravitational Potential at a point is defined as the work done by the external agent in bringing unit test mass from infinity to that point.

ie $V = \frac{W}{m_0}$, where V = gravitational Potential
W = work done
m_0 = test mass

Relation between Gravitational field & Potential



Let us suppose a point P at a distance r from an arbitrary origin O where the gravitational field and Potential are E and V respectively. Let Q be a point very close to P at a distance r+Δr where field and Potential are E+ΔE and V+ΔV respectively.

Naturally Potential difference between P and Q
$= V - (V + \Delta V) = -\Delta V =$ work done
by external agent in bringing from Q to P.
The external agent must apply a force opposite to the gravitational force. Assuming the force at P to Q be uniform, we have

$$-\Delta V = E \cdot \Delta r$$

or $E = -\frac{\Delta V}{\Delta r}$. This is only possible if Q is very close to P

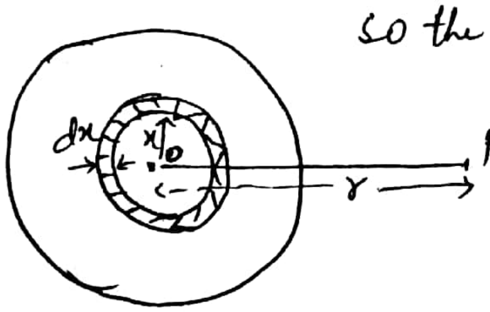
then $E = \lim_{\Delta r \to 0} -\frac{\Delta V}{\Delta r} = -\frac{dV}{dr}$ (ie -ve potential gradient)

$\boxed{\therefore E = -\frac{dV}{dr}}$ This is the desired relation.

Gravitational Potential and Field at a point due to a solid sphere

Let us consider a solid sphere of mass M and radius R. P is a point at a distance r from the centre of the solid sphere. To calculate potential at P, the sphere be divided into concentric shells of ~~dx~~ thickness dx. Let us take one of them of radius x. The volume of the taken shell $= 4\pi x^2 dx$

so the mass of the shell = volume × density



ie $dm = 4\pi x^2 dx \times \frac{M}{\frac{4}{3}\pi R^3}$

$$= \frac{3M}{R^3} x^2 dx$$

The potential at P due to this shell is
$dV = -\frac{G\,dm}{r}$ if $x < r$ and $dV = -\frac{G\,dm}{x}$ where $x > r$

Case I Potential at an external Point

When P is outside, the Potential at P is

$$= -G\frac{dm}{r}$$

Thus the Potential due to whole sphere is

$$V = \int dV = -\frac{G}{r}\int dm = -\frac{GM}{r}$$

$$\therefore \boxed{V = -\frac{GM}{r}}$$

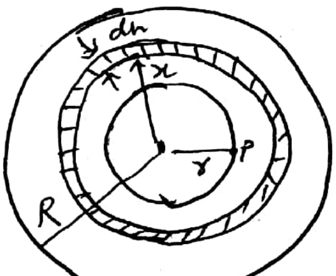
The Gravitational Potential due to uniform solid sphere at an external point is same as that due to a single particle of mass placed at its Centre.

The Gravitational field is

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2}$$

Case II When P lies inside the sphere

Let us Consider a Concentric spherical ~~shell~~ surface through P. The Potential at P is due to inner sphere and outer thick shell.



The mass of inner Sphere $= \frac{4}{3}\pi r^3 \rho$ where ρ = density of the sphere

The Potential at P due to this sphere is

$$V_1 = \frac{-G.\frac{4}{3}\pi r^3 \rho}{r} = -\frac{4}{3}\pi r^2 G\rho$$

To get the Potential at P due to outer thick shell, we consider this part in Concentric shell. Let us take one of them of radius x and thickness dx.

Its mass is $dm = 4\pi x^2 dx \rho$

The Potential at P due to this shell is

$$dV_2 = -G\frac{dm}{x} = -G.\frac{4\pi x^2 dx \rho}{x}$$

$$= -G \cdot 4\pi x dx \rho$$

The Potential due to whole thick shell is

$$V_2 = \int dV_2 = -G \cdot 4\pi\rho \int_r^R x dx = -G 4\pi\rho \left[\frac{x^2}{2}\right]_r^R$$

$$= -G 4\pi\rho \left(\frac{R^2 - r^2}{2}\right)$$

The Potential at P due to whole solid sphere

$$V = V_1 + V_2 = -G\frac{4}{3}\pi r^2 \rho - G 4\pi\rho\left[\frac{R^2 - r^2}{2}\right]$$

$$= -G 4\pi\rho\left[\frac{r^2}{3} + \frac{R^2 - r^2}{2}\right]$$

$$= -G\frac{4}{3}\pi\rho\left[\frac{3R^2 - r^2}{2}\right]$$

$$= -G \cdot \frac{4}{3}\pi R^3 \rho\left[\frac{3R^2 - r^2}{2R^3}\right] = -\frac{GM}{2R^3}\left[3R^2 - r^2\right]$$

$$\therefore V = -\frac{GM}{2R^3}\left[3R^2 - r^2\right]$$

Therefore, the gravitational field inside the sphere is

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left[-\frac{GM}{2R^3}(3R^2 - r^2)\right]$$

$$= -\frac{GM}{R^3} \cdot r$$

$$\boxed{\therefore E = -\frac{GM}{R^3} r}$$

The Potential at P due to this shell is

$$dV_2 = -G\frac{dm}{x} = -G.\frac{4\pi x^2 dx\,\rho}{x}$$

$$= -G.4\pi x\,dx\,\rho$$

The Potential due to whole thick shell is

$$V_2 = \int dV_2 = -G.4\pi\rho\int_r^R x\,dx = -G4\pi\rho\left[\frac{x^2}{2}\right]_r^R$$

$$= -G4\pi\rho\left(\frac{R^2-r^2}{2}\right)$$

The Potential at P due to whole solid sphere

$$V = V_1+V_2 = -G\frac{4}{3}\pi r^2\rho - G4\pi\rho\left[\frac{R^2-r^2}{2}\right]$$

$$= -G4\pi\rho\left[\frac{r^2}{3}+\frac{R^2-r^2}{2}\right]$$

$$= -G\frac{4}{3}\pi\rho\left[\frac{3R^2-r^2}{2}\right]$$

$$= -G.\frac{4}{3}\pi R^3\rho\left[\frac{3R^2-r^2}{2R^3}\right] = -\frac{GM}{2R^3}\left[3R^2-r^2\right]$$

$$\therefore V = -\frac{GM}{2R^3}\left[3R^2-r^2\right]$$

Therefore, the gravitational field inside the sphere is

$$E = -\frac{dv}{dr} = -\frac{d}{dr}\left[-\frac{GM}{2R^3}(3R^2-r^2)\right]$$

$$= -\frac{GM}{R^3}.r$$

$$\boxed{\therefore E = -\frac{GM}{R^3}r}$$

Gravitational Field and Potential

It is quite evident from Newton's law of gravitation that every body attracts another body. Naturally, the space surrounding the body up to where, the force of attraction effectively works, is called gravitational field.

It is measured by a term called <u>Intensity of Gravitational Field</u>. It is defined as the force experienced by unit test placed at that point.

ie. $\vec{E} = \dfrac{\vec{F}}{m_0}$ where $\vec{F}$ is the force

m_0 is the test mass

& $\vec{E}$ = Intensity

If $m_0 = 1$, the $\vec{E} = \vec{F}$

Gravitational Potential

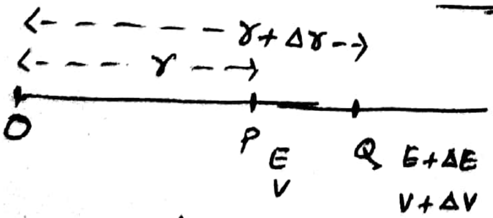
The gravitational potential at a point is defined as the work done by the external agent in bringing unit test mass from infinity to that point.

ie. $V = \dfrac{W}{m_0}$, where V = gravitational Potential

W = work done

m_0 = test mass

Relation between Gravitational field & Potential



Let us suppose a point P at a distance r from an arbitrary origin O where the gravitational field and Potential are E and V respectively. Let Q be a point very close to P at a distance $r+\Delta r$ where field and Potential are $E+\Delta E$ and $V+\Delta V$ respectively.