Theorem - Every bilinear form on a vector space $V(F)$ can be uniquely expressed as the sum of a symmetric and skew symmetric bilinear forms on $V(F)$

Proof: - Let f be a bilinear form on a vector space $V(F)$ and $u, v \in V$.

We can write

$$f(u,v) = \frac{1}{2}[f(u,v) + f(v,u)] + \frac{1}{2}[f(u,v) - f(v,u)] \quad \text{——— (i)}$$

let
$$\left.\begin{aligned} g(u,v) &= \frac{1}{2}[f(u,v) + f(v,u)] \\ h(u,v) &= \frac{1}{2}[f(u,v) - f(v,u)] \end{aligned}\right\} \text{—— (ii)}$$

Then $f(u,v) = g(u,v) + h(u,v) = (g+h)(u,v)$

This $f = g+h$ ——— (iii)

It is easy to verify that g and h both are bilinear forms on V.

from (ii)

$$g(v,u) = \frac{1}{2}[f(v,u) + f(u,v)] = g(u,v)$$

$$h(v,u) = \frac{1}{2}[f(v,u) - f(u,v)]$$

$$h(v,u) = -\frac{1}{2}[f(u,v) - f(v,u)] = -h(u,v)$$

Thus
$$\left.\begin{aligned} g(u,v) &= g(v,u) \\ h(u,v) &= -h(v,u) \end{aligned}\right\} \text{—— (iv)}$$

This proves that g is symmetric and h is anti-symmetric bilinear forms.

In this, Eq^n (iii) proves that the bilinear form f is ~~symmetric~~ sum of two bilinear forms, one of which is symmetric, while the other is anti-symmetric.

Remains to prove the uniqueness of representa-tion of f.

Let $f = f_1 + f_2$ be another representation of f, where f_1 is symmetric and f_2 is anti-symmetric.

Now $f(u,v) = f_1(u,v) + f_2(u,v)$ — ⑤ from ③

$f(v,u) = f_1(v,u) + f_2(u,v)$

$f(v,u) = f_1(v,u) + f_2(v,u)$ for symmetric $f(u,v) = f(v,u)$

$f(v,u) = f_1(u,v) - f_2(u,v)$ — ⑥ for Antisymm. $f(u,v) = -f(v,u)$

Adding and substracting ⑤ and ⑥ we get

Adding ④ & ⑥

$$f(u,v) + f(v,u) = 2 f_1(u,v)$$

$$f_1(u,v) = \frac{1}{2}[f(u,v) + f(v,u)]$$

$$g(u,v) = \frac{1}{2}[f(u,v) + f(v,u)]$$

Subtracting (4) and ⑥

$$f(u,v) - f(v,u) = 2 f_2(u,v)$$

$$f_2(u,v) = \frac{1}{2}[f(u,v) - f(v,u)]$$

$$h(u,v) = \frac{1}{2}[f(u,v) - f(v,u)]$$

This gives $f_1 = g$ and $f_2 = h$

Hence the uniqueness.