* **Bernoulli's Equation** –

The equation form as:-

$$\frac{dy}{dx} + Py = Qy^n \quad \text{—— (1)}$$

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is called the Bernoulli's Equation.

If $n = 0, 1$ then equation is linear, otherwise non-linear differential equation.

To solve Equation (1), divided both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{—— (2)}$$

Put $y^{1-n} = z$

$$(1-n)y^{-n}dy = dz$$

∴ Eqn (2) becomes

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

This is Leibnitz's linear Equation so we can solve easily.

Exp. Solve $x\frac{dy}{dx} + y = x^3y^5$

Solution Given Eqn divided by xy^5 both sides

$$y^{-5}\frac{dy}{dx} + \frac{y^{-4}}{x} = x^2 \quad \text{---(1)}$$

Put $y^{-4} = z$ then differenciate w.r.t.x

$$-4y^{-5}\frac{dy}{dx} = dz/dx$$

$$y^{-5}\frac{dy}{dx} = -\frac{1}{4}\frac{dz}{dx} \text{ put in Eqn(1)}$$

Then $$-\frac{1}{4}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$$\frac{dz}{dx} - \frac{4}{x}z = -4x^2$$

This is Leibnitz's Linear Equation in z.

$$I.F = e^{\int p\,dx} = \left[e^{-\int \frac{4}{x}dx}\right]$$

$$I.F = e^{-4\log x} = e^{\log x^{-4}} = x^{-4}$$

$$z.(I.F) = \int(-4x^2).(I.F)\,dx + c$$

$$z.x^{-4} = \int -4x^2.(x^{-4})\,dx + c$$

$$= -4\int x^{-2}dx + c$$

$$= +4.\frac{x^{-1}}{-1} + c$$

$$z = 4x^3 + cx^4$$

$$y^{-4} = 4x^3 + cx^4$$

$$1 = 4x^3y^4 + cx^4.y^4$$

* Exact Differential Equation :-

If $u(x,y)$ has a continuous partial derivatives then its total (exact) differential called exact differential of $u(x,y)$ i.e

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy \quad \text{—(I)}$$

A first order differential Equation

$$du = M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—(II)}$$

is called exact differential equation if the differential form $M(x,y)\,dx + N(x,y)\,dy$ has the differential as.

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

From (I) & (II)

$$\Rightarrow M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y} \text{ for some function } u(x,y)$$

Then equality $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is fulfilled, so equation (II) may be written in the form $du(x,y) = 0$ and is called exact differential equation.

* The solution of $M(x,y)\,dx + N(x,y)\,dy = 0$

$$\int_{(y \text{ constant})} M(x,y)\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

prodided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Exp. $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Solution.

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$$

$$M\,dx + N\,dy = 0$$

Here $M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus the equation is exact and its solution

$$\int_{(y\text{ const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int (y\cos x + \sin y + y)\,dx + \int (0)\,dx = c$$

$$y\int \cos x + \sin y \int dx + y\int dx = c$$

$$y\sin x + \sin y \cdot x + y \cdot x = c$$

$$y\sin x + (\sin y + y)x = c$$

This is the solution of the equation.