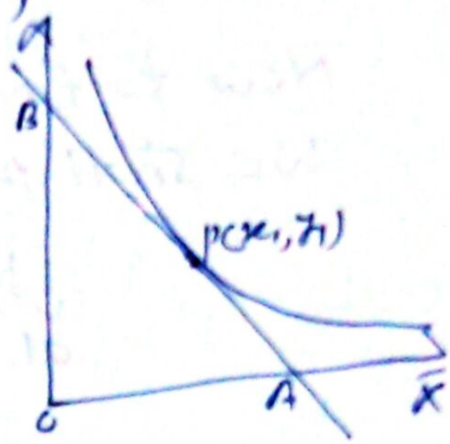
05/05/2021 TDC-I

Ex. Show that the portion of the tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ which intercepted between the axes is of constant length and find the area of the portion included between the axes and the tangent.



Sol. Given the equation of curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$

Diff. w.r.t. x, we get

$$\frac{2}{3}x^{\frac{2}{3}-1}+\frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx}=0$$

$$x^{-1/3}+y^{-1/3}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

$\therefore$ at point (x_1,y_1), $\left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}=-\frac{y_1^{1/3}}{x_1^{1/3}}$

The equation of the tangent at the point (x_1,y_1) is

$$y-y_1=-\frac{y_1^{1/3}}{x_1^{1/3}}(x-x_1)$$

$$x_1^{1/3}y-x_1^{1/3}y_1=-y_1^{1/3}x+y_1^{1/3}x_1$$

$$x_1^{1/3}y+y_1^{1/3}x=x_1^{1/3}y_1+y_1^{1/3}x_1$$

$$x_1^{1/3}y+y_1^{1/3}x=x_1^{1/3}y_1^{1/3}\left(y_1^{2/3}+x_1^{2/3}\right) \quad (1)$$

$$x_1^{1/3}y+y_1^{1/3}x$$

Since (x_1, y_1) lies on the curve then

$$x_1^{2/3} + y_1^{2/3} = a^{2/3}$$

putting in Eqn (I)

$$x_1^{1/3} y + y_1^{1/3} x = x_1^{1/3} y_1^{1/3} . a^{2/3} \text{ ——— (II)}$$

Now to find out the intercept on the x axis we shall put $y=0$ in Eqn (II)

$$y_1^{1/3} x = x_1^{1/3} y_1^{1/3} a^{2/3}$$

$$x = x_1^{1/3} a^{2/3}$$

$\therefore$ the intercept of the tangent on the x-axis $OA = x_1^{1/3} . a^{2/3}$, which the tangent cuts the x-axis on $OA = x_1^{1/3} a^{2/3}$

Similarly, the intercept of the tangent on the y-axis i.e $x=0$ put in Eqn (II) then y co-ordinate of the point at which the tangent cuts the y-axis on $OB = y_1^{1/3} a^{2/3}$

Thus the co-ordinate of

$$A = (x_1^{1/3} a^{2/3}, 0)$$

$$B = (0, y_1^{1/3} a^{2/3})$$

Now $(AB)^2 = (x_1^{1/3} a^{2/3} - 0)^2 + (0 - y_1^{1/3} a^{2/3})^2$

$$(AB)^2 = x_1^{2/3} . a^{4/3} + y_1^{2/3} a^{4/3}$$

$$(AB)^2 = a^{4/3}\left[x_1^{2/3} + y_1^{2/3}\right]$$

$$= a^{4/3} \cdot a^{2/3}$$

$$(AB)^2 = a^2$$

$$AB = a = \text{constant.}$$

Now find out the area of ΔOAB

$$= \frac{1}{2} OA \times OB$$

$$= \frac{1}{2} x_1^{1/3} a^{2/3} \cdot y_1^{1/3} \cdot a^{2/3}$$

$$= \frac{1}{2} a^{4/3} \cdot (x_1 \cdot y_1)^{1/3}$$