Analytical geometry of three dimensions

The plane :-

Q.) Find the condition that the homogeneous equation of the second degree i.e. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may represent a pair of planes.

Also obtain the angle between the planes.

Ans:- Let the homogeneous equation of the second degree be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{—①}$$

Let the two planes represented by ① be

$$l_1x+m_1y+n_1z=0 \quad \text{and} \quad l_2x+m_2y+n_2z=0$$

as the given equation is homogeneous, so there will be no constant terms in the seperate equations of the planes.

$$\therefore ax^2+by^2+cz^2+2fyz+2gzx+2hxy \equiv (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)$$

Comparing the co-efficients of like terms we get

$$a=l_1l_2, \quad 2f=m_1n_2+m_2n_1$$
$$b=m_1m_2, \quad 2g=n_1l_2+n_2l_1$$
$$c=n_1n_2 \quad 2h=l_1m_2+l_2m_1$$

Now $8fgh = (m_1n_2 + m_2n_1)(n_1l_2 + n_2l_1)(l_1m_2 + l_2m_1)$

$= 2l_1l_2m_1m_2n_1n_2 + l_1l_2(m_1^2n_2^2 + m_2^2n_1^2)$
$+ m_1m_2(n_1^2l_2^2 + n_2^2l_1^2) + n_1n_2(l_1^2m_2^2 + l_2^2m_1^2)$

$= 2abc + a[(m_1n_2 + m_2n_1)^2 - 2m_1m_2n_1n_2]$
$+ b[(n_1l_2 + n_2l_1)^2 - 2n_1n_2l_1l_2] + c[(l_1m_2 + l_2m_1)^2$
$- 2l_1l_2m_1m_2]$

$= 2abc + a[4f^2 - 2bc] + b[4g^2 - 2ca] + c[4h^2 - 2ab]$

i.e. $4fgh = abc + 2af^2 - abc + 2bg^2 - abc + 2ch^2 - abc$

i.e. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

This is the required Condition.

Let θ be the angle between the pair of planes.

Then $\tan\theta = \dfrac{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}$

$= \dfrac{\sqrt{(m_1n_2 + m_2n_1)^2 - 4m_1m_2n_1n_2 + (n_1l_1 + n_2l_2)^2 - 4n_1n_2l_1l_2 + (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}{a + b + c}$

$= \dfrac{\sqrt{(4f^2 - 4bc) + (4g^2 - 4ca) + (4h^2 - 4ab)}}{a + b + c}$

$= \dfrac{2\sqrt{f^2 + g^2 + h^2 - bc - ca - ab}}{a + b + c}$

Note :- The planes will be at right angles if $a + b + c = 0$.