Viscosity.

Poiseuille's formula for flow of a liquid through a narrow tube :-

Consider a cylindrical horizontal capillary tube of radius 'a' and length 'l' through which a liquid is flowing under a constant pressure difference P applied between the ends of the tube. It is assumed that

(i) the flow of liquid is stream lined and everywhere parallel to the axis of the tube,

(ii) the flow is steady, i.e. there is no acceleration of the liquid at any point;

(iii) there is no radial flow so that from Bernoulli's theorem, the pressure will be constant over any cross-section

(iv) the velocity of the layer of liquid in contact with the walls of the tube is zero and increases regularly and continuously towards the interior becoming maximum along the axis of the tube.

Velocity of flow in any layer:

Let us consider a cylindrical layer of the liquid coaxial with the tube and of radius x. Its cylindrical surface area is equal to $2\pi xl$ and the velocity of flow at all the points on this cylindrical layer is the same. But since the velocity of flow increases from the wall towards the axis of the tube, the liquid outside the cylindrical layer will be moving slower than that inside it and hence will exert a backward tangential force on the inner fast moving liquid, the retarding viscous force, in accordance to Newton's law of viscous flow is given by

$$F = -\eta A \frac{dv}{dx}$$

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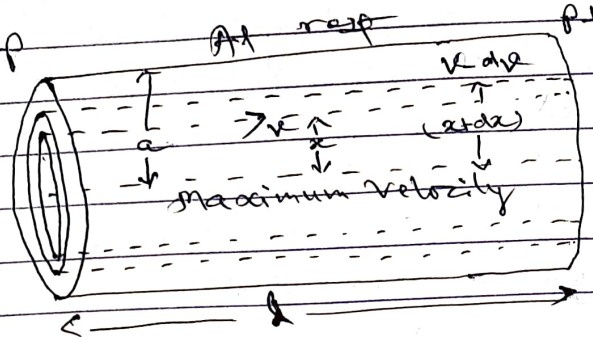
$$F = -\eta\, 2\pi x l \frac{dv}{dx} \quad \text{——— (1)} \qquad (\because A = 2\pi x l)$$



where η is the coefficient of viscosity of the liquid, v is the velocity of flow at a distance x from the axis and $\frac{dv}{dx}$ is the velocity gradient there.

Now the flow of liquid is maintained by a forward force due to the difference of pressure p between the two ends of the tube. Hence the force tending to accelerate the cylindrical liquid layer

= pressure difference × area of cross-section

$= p \times \pi x^2$ ——— (2)

As the velocity of liquid layer is constant, the acceleration and retarding forces on the layer will be equal. Equating (1) and (2) we get, therefore,

$$-\eta\, 2\pi x \cdot l \frac{dv}{dx} = \pi x^2 \times p$$

$$dv = -\frac{px}{2\eta l}\, dx.$$

Integrating this expression we get

$$v = -\frac{px^2}{2\eta l \cdot 2} + C \quad \text{——— (3)}$$

where C is constant of integration.

Now the liquid in contact with the walls of the tube is at rest i.e. $v = 0$, when $x = a$. Hence

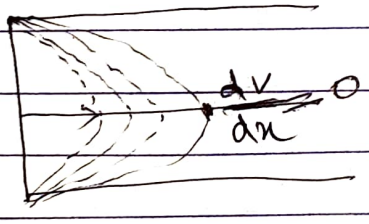
$$0 = -\frac{pa^2}{2\eta l \cdot 2} + C$$

or $$C = \frac{pa^2}{4\eta l}$$

Equation (3) thus becomes

$$v = -\frac{px^2}{4\eta l} + \frac{pa^2}{4\eta l}$$

$$= \frac{p}{4\eta l}(a^2 - x^2) \quad \text{—— (4)}$$



This gives the velocity of flow at a distance x from the axis of the tube and shows that the profile of advancing liquid is a parabola as shown in fig.

Volume of liquid flowing out:

If we consider another coaxial layer of radius $x + dx$, then the velocity of flow between the two cylindrical layers is v and $2\pi x\,dx$ is the cross-sectional area between them, volume of liquid flowing per second through this area,

dV = Velocity × Cross-Sectional area

$= v \times 2\pi x\, dx$

$\frac{dV}{} = \frac{p}{4\eta l}(a^2 - x^2) \times 2\pi x\, dx.$

Now the bore of tube may be imagined to be made of a large number of such sections and the total volume V of liquid flowing per second through the tube may be obtained by integrating the above expression between the limits $x = 0$ (at axis) and $x = a$ (at the walls). Hence

$$V = \int_0^a \frac{p}{4\eta l}(a^2 - x^2) \times 2\pi x\, dx$$

$$= \frac{2\pi p}{4\eta l}\int_0^a (a^2 x - x^3)\, dx$$

$$= \frac{\pi p}{2\eta l}\left[a^2 \cdot \frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{\pi p}{2\eta l}\left[a^2 \cdot \frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{\pi p a^4}{8\eta l} \qquad \text{——— (5)}$$

This is Poiseuille's formula for the rate of flow of liquid Through a capillary Tube.

Limitations and Corrections to Poiseuille's formula:

(A) Limitations: (i) The formula applies only to streamline flow through the tube. The flow is streamline when the velocity of flow is less than the critical velocity. Since critical velocity of liquid is inversely proportional to radius of the tube, this flow will tend to become turbulent in case of tubes of wide bore. Thus Poiseuille's formula holds good for narrow tubes only.

(ii) The formula breaks down if the liquid layers in contact with the wall are not stationary. For it the pressure difference across the capillary tube should be kept low so that liquid flows very slowly through the tube.

(iii) The Poiseuille's formula holds good only so long as the tube is horizontal and escaping fluid has negligible K.E.

(iv) Poiseuille's formula is not valid for gases.

(B) Corrections (i) The flow has been assumed to be steady and uniform throughout the length of the tube. In actual practice, the flow of liquid is slightly accelerated at the entrance and becomes uniform after traversing a small length of the tube. This error is corrected by taking the effective length of the tube as $(l + 1.64a)$ in place of l, where a is the radius of the tube.

ii) It is further assumed that the applied pressure difference between the ends of the flow tube is entirely used in overcoming viscous resistance. In actual practice, a part of it is used in imparting K.E. to the liquid leaving the tube. To correct for this error, the effective pressure difference between the ends of the tube should be taken as

$\left(P - \frac{V^2 \rho}{2}\right)$ where ρ is the density of the liquid.